A path integral approach to Asset-Liability Management

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Abstract

Functional integrals constitute a powerful tool in the investigation of financial models. In the recent econophysics literature, this technique was succesfully used for the pricing of a number of derivative securities. In the present contribution, we introduce this approach to the field of asset-liability management. We work with a representation of cash flows by means of two-dimensional delta-function perturbation, in the case of a Brownian model and a geometric Brownian model. We derive closed-form solutions for a finite horizon ALM policy. The results are numerically and graphically illustrated.

Keywords: functional integral, ALM, δ-function perturbation, local time, spectral method.

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1 Introduction

Banks, insurance companies and other corporations have to report on liabilities and assets to their shareholders and creditors. As an example, traditionally in insurance business, liabilities take on proceeds from deposits, life insurance policies, annuities, etc. The cash flows from liabilities are then invested in assets such as equities, fixed income instruments, real estates, etc. As far as market risk is concerned, Value-at-Risk (VaR) and more generally all types of risk measures have proven to be very suitable techniques to control the solvency of the company. On the other hand, financial firms are increasingly forced to use market-value accounting for certain business lines, including trading books of listed derivatives. For business lines accounted on an accrual basis (it includes some of the most traditional insurance activities), however, a dynamic match between the assets and the liabilities is much more appropriate. The stochastic control of the balance sheet is achieved by Asset-Liability Management (ALM) techniques. ALM techniques consist of controlling the positive and negative cash flows generated by the company activity in order to narrow the difference between assets and liabilities while maximizing the creation of value. The question of such a dynamic approach is important; indeed, it is not only true that some of the most traditional insurances activities belong to this group, but the methodology will also be soon required by the regulator.

This paper is an attempt to contribute to the growing research field made up by ALM using techniques arising in quantum mechanics, more precisely, Brownian functionals integration. Functional integration techniques and, more generally, the path integral formalism have been succesfully applied for financial challenges, especially in the context of derivative pricing. As an example, we can refer to contributions in econophysics of e.g. Baaquie (1997), Chiarella et al. (1999), Kleinert (2002), Matacz (2002), Montagna et al. (2002a, 2002b, 2003) and Rosa-Clot & Taddei (2002)[1].

ALM has received considerable attention for the last decades in the more traditional actuarial and financial literature. Without claiming exhaustiveness, we cite Asmussen & Taksar (1997), Hojgaard & Taksar (1999), Taksar (2000), Hojgaard (2002) and Gerber & Shiu (2004). Asmussen & Taksar (1997) investigate the maximisation of the discounted average total pay-out of dividends up to the time of ruin. They consider a Brownian

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[1] Basic information about these techniques can be found for example in Feynman & Hibbs (1965), Ito & McKean (1965), and Revuz & Yor (1991)

motion with drift for the company surplus, and they look for an optimal dividend payment scheme. Hojgaard & Taksar (1999) give further results, and determine an optimal proportional reinsurance program in the same diffusion model. Additional results can be found in Taksar (2000) and Hojgaard (2002). Gerber & Shiu (2004) model assets and liabilities by means of geometric Brownian motions and investigate surplus in an infinite time-horizon.

In the present paper, we contribute to the same research field of dynamic control; the use of functional integration enables us to derive analytical results in a finite time-horizon. In order to do so, δ-function perturbation turns out to be a suitable technique. In this context, interesting references in the field of mathematical physics are from Goovaerts et al. (1973), Grosche (1990, 1993) and Carreau (1992); in the field of probability theory we can mention papers of Dai & Harrison (1991, 1992) and Burdzy & Nualart (2002).

Two models will be investigated. In order to introduce the path integral formalism in ALM, we start to describe the asset and liability values of the firm by means of a two-dimensional Brownian motion with drift. In most contributions about dynamic control, this type of process serves as a (first) approximation. The value of the firm is then modeled as $\boldsymbol{F}(t) = (A(t), L(t))$, with $i = 1, 2$

$$dF_i(t) = \mu_i dt + \sigma_i dB_i(t) \tag{1}$$

or

$$F_i(t) = F_i(0) + \mu_i t + \sigma_i B_i(t) \tag{2}$$

where $\boldsymbol{F}(0) = (a, l)$, and where $B_1(t)$ and $B_2(t)$ are correlated standard Brownian motions (correlation ρ). For later use, we define $\boldsymbol{\mu} = (\mu_1, \mu_2)$, $\boldsymbol{B}(t) = (B_1(t), B_2(t))$ and the matrix

$$\boldsymbol{\sigma}^2 := \left(\sigma_{i,j}^2\right) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

In the whole contribution, the index 1 refers to the assets $A(t)$ (e.g. μ_1, $\boldsymbol{B}_1$, F_1, ...), while the index 2 refers to the liabilities $L(t)$.

The ALM policy consists of maintaining the time-value of the firm in a corridor (or Allowable set)

$$\begin{aligned}
\mathcal{A}(\lambda_1,\lambda_2) &= \{\boldsymbol{x}=(x_1,x_2)\in\mathbb{R}^2 |\ x_1 \geq x_2+\lambda_1 \ \mathit{and}\ x_1\leq x_2+\lambda_2\} \\
e(\lambda_1) &= \{\boldsymbol{x}=(x_1,x_2)\in\mathbb{R}_2 |\ x_1 = x_2+\lambda_1\} \\
e(\lambda_2) &= \{\boldsymbol{x}=(x_1,x_2)\in\mathbb{R}_2 |\ x_1 = x_2+\lambda_2\}.
\end{aligned}$$

The set $\partial\mathcal{A} = e(\lambda_1)\cup e(\lambda_2)$ is the frontier of the allowable set and $\mathcal{B} = \mathbb{R}^2\backslash\mathcal{A}$ is the Bankruptcy set. We further assume that the ALM committee is subject to some predefined guidelines:

- the ALM committee is only allowed to act when it is strictly necessary, namely when the firm value is about to exit the set $\mathcal{A}$.
- The ALM committee can act on both the assets and the liabilities but in a given proportion η_1/η_2 (a change of $\Delta A(t)$ in the assets implies a change $\Delta L(t) = \frac{\eta_1}{\eta_2}\Delta A(t)$ in the liabilities). The ratio η_1/η_2 may a priori depend on the firm value. It is clear that η_1 is negative at the upper barrier (e.g. through the payment of dividend to shareholders) and η_2 is positive (e.g. through investment into new lines of business). While at the lower barrier, η_1 is positive (e.g. through fresh capital) and η_2 is negative (e.g. by selling out parts of the business through reinsurance, OTC derivatives, excesses, etc).
- The cashflows are unbounded. This assumption is of course not realistic but it provides a tractable approximation.

The ALM strategy that fulfills all the requirements is achieved by imposing oblique reflection on $\partial\mathcal{A}$ along the reflection (unit) vector $\boldsymbol{\eta}(\boldsymbol{x}) = (\eta_1(\boldsymbol{x}),\eta_2(\boldsymbol{x}))$. We assume $\boldsymbol{\eta}(\boldsymbol{x})$ to be constant over $e(\lambda_1)$ and $e(\lambda_2)$; we then use the notation $\boldsymbol{\eta}^{(\lambda_i)}$ for the reflection vector along $e(\lambda_i)$, $i=1,2$. If e.g. we work with $\boldsymbol{\eta}^{(\lambda_2)} = (-1,0)$, this means that at the upper barrier, if the firm value would leave the allowable set, we act on the assets and not on the liabilities, e.g. by the payment of dividends to shareholders.

Cash flows occur only on the frontier $\partial\mathcal{A}$ and the modified time-value of the firm is a two-dimensional η-reflected Brownian motion inside $\mathcal{A}$ and follows the diffusion equations, with $i=1,2$,

$$dF_i(t) = \mu_i dt + \sigma_i dB_i(t) + \frac{1}{2}\eta_i(\boldsymbol{F}(t))dL_t(\boldsymbol{F}) \quad (i=1,2) \tag{3}$$

where $L_t(\boldsymbol{F})$ is the local time of the diffusion $\boldsymbol{F}$ along the frontier $\partial\mathcal{A}$ defined by

$$L_t(\boldsymbol{F}) = L_t^{\lambda_1,+}(\boldsymbol{F}) + L_t^{\lambda_2,-}(\boldsymbol{F}) \tag{4}$$

where $L_t^{\lambda_i,\pm}(F)$ $(i = 1, 2)$ measures the time spent in the vicinity of the edge $e(\lambda_i)$ up to time t and is defined by the formulae (to some normalizing factor)

$$\begin{aligned} L_t^{\lambda_1,+}(\boldsymbol{F}) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t 1_{(0,\epsilon)}(A(s)-L(s)-\lambda_1)ds, \quad \text{(right local time)}\\ L_t^{\lambda_2,-}(\boldsymbol{F}) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t 1_{(-\epsilon,0)}(A(s)-L(s)-\lambda_2)ds, \quad \text{(left local time).} \end{aligned}$$

See e.g. Carreau (1992) in mathematical physics and Dai & Harrison (1991, 1992) in probability theory. The total value creation at the upper barrier is thus

$$C_t^{(2)}(\boldsymbol{F}) = \frac{1}{2}\sum_{i=1}^{2}\int_0^t e^{-rs}|\eta_i^{(\lambda_2)}|dL_s^{\lambda_2,-}(\boldsymbol{F}), \tag{5}$$

where r is the constant force of interest rate.

Note that the ALM strategy is fully determined by $A/LM := (\mathcal{A}, \boldsymbol{\eta}(\boldsymbol{F}))$.

A more realistic model will involve a geometric Brownian motion in order to preclude non-positive values for the assets and the liabilities. The time-value of the firm now is $\boldsymbol{F}(t) = (A(t), L(t))$, with $i = 1, 2$,

$$\frac{dF_i(t)}{F_i(t)} = \left(\mu_i + \frac{1}{2}\sigma_i^2\right)dt + \sigma_i dB_i(t) \tag{6}$$

or

$$F_i(t) = F_i(0)\ \exp\{\mu_i t + \sigma_i B_i(t)\} \tag{7}$$

where as before $\boldsymbol{F}(0) = (a, l)$, $B_1(t)$ and $B_2(t)$ are correlated standard Brownian motions (correlation ρ).

Through the transformation $exp\{\boldsymbol{x}\}$, the allowable set $\mathcal{A}$ becomes the following cone

$$\mathcal{A}(\lambda_1, \lambda_2) = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 |\ x_1 \geq \lambda_1 x_2\ and\ x_1 \leq \lambda_2 x_2\}.$$

and the edges, the lines

$$e(\lambda_i) = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}_2 |\ x_1 = \lambda_i\ x_2\}.$$

The modified time-value of the firm inside $\mathcal{A}$ is a two-dimensional η-reflected geometric Brownian motion driven by the modified diffusion equations, with $i = 1, 2$,

$$dF_i(t) = \left(\mu_i + \frac{1}{2}\sigma_i^2\right) F_i dt + \sigma_i F_i dB_i(t) + \frac{1}{2}\eta_i(\boldsymbol{F}(t)) dL_t(\boldsymbol{F}) \quad (i = 1, 2) \tag{8}$$

where $L_t(\boldsymbol{F})$ is the local time of the diffusion $\boldsymbol{F}$ along the frontier $\partial\mathcal{A}$ defined by

$$L_t(\boldsymbol{F}) = L_t^{\lambda_1,+}(\boldsymbol{F}) + L_t^{\lambda_2,-}(\boldsymbol{F}) \tag{9}$$

with

$$\begin{aligned} L_t^{\lambda_1,+}(\boldsymbol{F}) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{(0,\epsilon)}(A(s) - \lambda_1 L(s)) ds, \quad \text{(right local time)} \\ L_t^{\lambda_2,-}(\boldsymbol{F}) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{(-\epsilon,0)}(A(s) - \lambda_2 L(s)) ds, \quad \text{(left local time)} \end{aligned}$$

to some normalizing factor. Again, the ALM strategy is fully determined by $A/LM := (\mathcal{A}, \boldsymbol{\eta}(\boldsymbol{F}))$. By standard use of Itô formula, we can check that

$$F_i(t) = F_i(0)\ exp\left\{\mu_i t + \sigma_i B_i(t) + \frac{\eta_i(\boldsymbol{F})}{2} \int_0^t \frac{dL_s(\boldsymbol{F})}{F_i(s)}\right\}.$$

As the local time of $\boldsymbol{F}$ on $e(\lambda_2)$ satisfies the identity

$$\begin{aligned} \int_0^t \frac{dL_s^{\lambda_2,-}(\boldsymbol{F})}{F_2(s)} &= \int_0^t 1_{\boldsymbol{F}(s) \in e(\lambda_2)} \frac{dL_s^{\lambda_2,-}(\boldsymbol{F})}{F_2(s)} \\ &= \lambda_2 \int_0^t \frac{dL_s^{\lambda_2,-}(\boldsymbol{F})}{F_1(s)}, \end{aligned} \tag{10}$$

we deduce that the process $\{\ln \boldsymbol{F}, t \geq 0\}$ is a Brownian motion with drift reflected on the edge

$$\ln e(\lambda) = \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 + \ln \lambda_2\},$$

along the direction indicated by the (to be normalized) vector $\left(\eta_1^{(\lambda_2)}, \lambda_2\eta_2^{(\lambda_2)}\right)$. Remark that the reflection vector remains unchanged only if either $\boldsymbol{\eta}^{(\lambda_2)} = (-1, 0)$, $\boldsymbol{\eta}^{(\lambda_2)} = (0, 1)$ or $\lambda_2 = 1$. Similar transformation holds at the lower edge $e(\lambda_1)$.

The paper is organized as follows. We start in section 2 with a (concise) summary of the most important technical definitions and results about functional integration, η-reflected paths and δ-function perturbation. In section 3, we explain how a functional integration can be used to describe the time-evolution of the cash flows generated by the ALM policy $(\mathcal{A}, \boldsymbol{\eta}(\boldsymbol{F}))$. The solution to these equations is given in a closed form in section 4, and illustrated numerically and graphically afterwards in section 5. Section 6 concludes.

2 Path integrals over η-reflected paths

A free particle moving inside the set $\mathcal{A}$ and oblique η-reflected on $\partial\mathcal{A} = e(\lambda_1) \cup e(\lambda_2)$ can be described by the self-adjoint operator

$$\frac{1}{2}\sum_{i,j=1}^{2} \sigma_{i,j}^2 \frac{d^2}{dx_i dx_j}$$

together with the following boundary conditions on $\partial\mathcal{A}$ for any function ϕ in the domain of the operator

$$\boldsymbol{\eta}(\boldsymbol{x}) \cdot \nabla\phi(\boldsymbol{x})|_{\boldsymbol{x}\in\partial\mathcal{A}} = 0 \tag{11}$$

where $\nabla = (d/dx_1, d/dx_2)^T$ is the gradient.
When the particle is subject to a potential $V(\boldsymbol{x}) = V(x_1, x_2)$, the amplitude to go from one point to another can be described by path integrals over η-reflected Brownian paths in the set $\mathcal{A}$. We define the Brownian path integral

$$I_V^\eta(\boldsymbol{x}, \boldsymbol{y}, t) = \int_{(0,\boldsymbol{x})}^{(t,\boldsymbol{y})} D\boldsymbol{x}_\eta(s) e^{-\frac{1}{2}\int_0^t \dot{\boldsymbol{x}}_\eta^2(s)ds - \int_0^t V(\boldsymbol{x}_\eta(s))ds}. \tag{12}$$

The symbol $D\boldsymbol{x}_\eta(s)e^{-\frac{1}{2}\int_0^t \dot{\boldsymbol{x}}_\eta^2(s)ds}$ where $\dot{\boldsymbol{x}}_\eta^2(s)$ holds for

$$\sigma_1^2\dot{x}_1^2(s) + \sigma_2^2\dot{x}_2^2(s) + 2\rho\sigma_1\sigma_2\dot{x}_1(s)\dot{x}_2(s), \quad \dot{x}_i(s) = \frac{d}{ds}x_i(s)$$

has to be interpreted as the Brownian measure that gives non-zero weight to continuous paths reflected inside the set $\mathcal{A}$ along the vector $\boldsymbol{\eta}(\boldsymbol{x})$ whenever the particle hits the frontier $\partial\mathcal{A}$. The Green function, which is the the Laplace transform of the path integral (12)

$$G_V^\eta(\boldsymbol{x}, \boldsymbol{y}, s) = \int_0^{+\infty} dt\ e^{-st} \int_{(0,\boldsymbol{x})}^{(t,\boldsymbol{y})} D\boldsymbol{x}_\eta(s) e^{-\frac{1}{2}\int_0^t \dot{\boldsymbol{x}}_\eta^2(s)ds - \int_0^t V(\boldsymbol{x}_\eta(s))ds},$$

satisfies the following integro equation

$$G_V^\eta(\boldsymbol{x}, \boldsymbol{y}, s) + \int G_0^\eta(\boldsymbol{x}, \boldsymbol{\xi}, s) V(\boldsymbol{\xi}) G_V^\eta(\boldsymbol{\xi}, \boldsymbol{y}, s) d\boldsymbol{\xi} = G_0^\eta(\boldsymbol{x}, \boldsymbol{y}, s) \tag{13}$$

where $G_0^\eta(\boldsymbol{x}, \boldsymbol{y}, s)$ is the Green function of the free (η-reflected) particle confined in $\mathcal{A}$. This can be shown by means of an expansion of the exponential function in (12) and the Kac's moments formula. As the Green function of the free particle satisfies

$$\frac{1}{2} \sum_{i,j=1}^{2} \sigma_{i,j}^2 \frac{d^2}{dx_i dx_j} G(\boldsymbol{x}, \boldsymbol{y}, s) = s\ G(\boldsymbol{x}, \boldsymbol{y}, s) - \delta(\boldsymbol{y} - \boldsymbol{x})$$

where $\delta(.)$ stands for the two-dimensional Dirac delta function[2], we conclude that the Green function of the particle restricted to the set $\mathcal{A}$ in the potential V is solution of the following two-dimensional Schrödinger equation in imaginary time

$$\left[-\frac{1}{2} \sum_{i,j=1}^{2} \sigma_{i,j}^2 \frac{d^2}{dx_i dx_j} + V(\boldsymbol{x}) \right] G(\boldsymbol{x}, \boldsymbol{y}, s) = -s\ G(\boldsymbol{x}, \boldsymbol{y}, s) + \delta(\boldsymbol{y} - \boldsymbol{x}), \tag{14}$$

subject to boundary conditions $\boldsymbol{\eta}(\boldsymbol{x}) \cdot \nabla G(\boldsymbol{x}, \boldsymbol{y}, s)|_{\boldsymbol{x} \in \partial\mathcal{A}} = 0$ for any $\boldsymbol{y}$ in the interior of $\mathcal{A}$ ($\boldsymbol{y} \neq \boldsymbol{x}$). By inversion of the Laplace transform, we conclude that the path integral (12) is solution of the Schrödinger equation (in imaginary time) with η-reflected condition on $\partial\mathcal{A}$:

$$\begin{cases} \frac{d}{dt} I(\boldsymbol{x}, \boldsymbol{y}, t) = \left[\frac{1}{2} \sum_{i,j=1}^{2} \sigma_{i,j}^2 \frac{d^2}{dx_i dx_j} - V(\boldsymbol{x}) \right] I(\boldsymbol{x}, \boldsymbol{y}, t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (\boldsymbol{x}, \boldsymbol{y}, t) \in \mathcal{A}^2 \times \mathbb{R}^+ \\ I(\boldsymbol{x}, \boldsymbol{y}, 0) = \delta(\boldsymbol{x} - \boldsymbol{y}), \quad (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}^2 \\ \boldsymbol{\eta}(\boldsymbol{x}) \cdot \nabla I(\boldsymbol{x}, \boldsymbol{y}, t) = 0, \quad (\boldsymbol{x}, \boldsymbol{y}, t) \in \partial\mathcal{A} \times \mathcal{A} \times \mathbb{R}^+. \end{cases}$$

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[2] $\delta(\boldsymbol{x}) = \delta(x_1)\delta(x_2)$ where $\delta(x)$ is the one-dimensional Dirac delta function, $\int dx_1 \int dx_2 f(x_1, x_2) \delta(\boldsymbol{x}) = f(0)$.

The path integral (12) is a sum over all possible trajectories of the quantity $\exp\{-\int_0^t V(\boldsymbol{x}(s))ds\}$. For ALM applications, we need to describe the time-evolution of cash flows that increase only when the firm value is about to leave the allowable set $\boldsymbol{\mathcal{A}}$. The trajectories of the firm value contribute to cash flows only near the frontier $\partial\mathcal{A}$. Therefore, we consider in what follows paths in the local time (measuring the time spent near the edge $e(\lambda_i)$)

$$\exp\left\{-\int_0^t \alpha(\boldsymbol{X}(s))dL_s^{\lambda_i}(\boldsymbol{X})\right\} = \exp\left\{-\int_0^t \tilde{V}_{\lambda_i}(\boldsymbol{X}(s))ds\right\}$$

where $\tilde{V}_{\lambda_i}(\boldsymbol{x}) = \alpha(\boldsymbol{x})\delta(x_2 - x_1 - \lambda_i)$ is a singular potential involving a Dirac δ- function. Note that $\tilde{V}_{\lambda_i}$ returns non-zero values only along the edge $e(\lambda_i)$. Inserting the potential $V_{\lambda_i}(\boldsymbol{x}) = V(\boldsymbol{x}) + \tilde{V}_{\lambda_i}(\boldsymbol{x})$ in the integro relation (13), we obtain that

$$\begin{aligned} G^{\eta}_{V_{\lambda_i}}(\boldsymbol{x}, \boldsymbol{y}, s) &= G^{\eta}_{V}(\boldsymbol{x}, \boldsymbol{y}, s) \\ &\quad - \int_{e(\lambda_i)} G^{\eta}_{V}(\boldsymbol{x}, \boldsymbol{\xi}, s)\alpha(\boldsymbol{\xi})G^{\eta}_{V_{\lambda_i}}(\boldsymbol{\xi}, \boldsymbol{y}, s)d\boldsymbol{\xi} \end{aligned}$$

or equivalently, the associated Green function $G^{\eta}_{V_{\lambda_i}}(\boldsymbol{x}, \boldsymbol{y}, s)$ satisfies the Schrödinger equation (14) subject to the condition

$$\frac{1}{2}\,\boldsymbol{\eta}^{(\lambda_i)} \cdot \nabla G(\boldsymbol{x}, \boldsymbol{y}, s)|_{\boldsymbol{x}\in e(\lambda_i)} = \alpha(\boldsymbol{x})G(\boldsymbol{x}, \boldsymbol{y}, s)|_{\boldsymbol{x}\in e(\lambda_i)}.$$

The boundary condition on $e(\lambda_i)$ results from the derivation with respect to $\boldsymbol{x}$ of equation (15) together with

$$\boldsymbol{\eta}^{(\lambda_i)} \cdot \nabla G^{\eta}_{V}(\boldsymbol{x}, \boldsymbol{y}, s)|_{\boldsymbol{x}\in e(\lambda_i)} = 0$$

except when $\boldsymbol{y} \in e(\lambda_i)$, where

$$\boldsymbol{\eta}^{(\lambda_i)} \cdot \nabla G^{\eta}_{V}(\boldsymbol{x}, \boldsymbol{y}, s) = -2\delta(x_2 - y_2), \quad \boldsymbol{x}, \boldsymbol{y} \in e(\lambda_i).$$

Path integrals over oblique reflected Brownian paths enable us to formulate the transition density of the two-dimensional Brownian motion with drift $\boldsymbol{\mu}$ η-reflected on $\partial\mathcal{A}$. The infinitesimal generator is the operator

$$\mathcal{G} = \frac{1}{2}\sum_{i,j=1}^{2} \sigma^2_{i,j}\frac{d^2}{dx_i dx_j} + \boldsymbol{\mu} \cdot \nabla$$

and the transition density $p_{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{y}, t)$ such that $\mathrm{Prob}[\boldsymbol{X}(t) \in d\boldsymbol{x} | \boldsymbol{X}(0)] = p_{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{y}, t) d\boldsymbol{x}$ satisfies the two-dimensional backward Fokker-Planck equation

$$\begin{cases} \frac{d}{dt} I(\boldsymbol{x}, \boldsymbol{y}, t) = \mathcal{G} I(\boldsymbol{x}, \boldsymbol{y}, t), & (\boldsymbol{x}, \boldsymbol{y}, t) \in \mathcal{A}^2 \times \mathbb{R}^+ \\ I(\boldsymbol{x}, \boldsymbol{y}, 0) = \delta(\boldsymbol{x} - \boldsymbol{y}), & (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}^2 \\ \boldsymbol{\eta}(\boldsymbol{x}) \cdot \nabla I(\boldsymbol{x}, \boldsymbol{y}, t) = 0, & (\boldsymbol{x}, \boldsymbol{y}, t) \in \partial\mathcal{A} \times \mathcal{A} \times \mathbb{R}^+. \end{cases}$$

Standard algebra[3] yields the following path integral for $p_X(x, y, t)$:

$$\begin{aligned} p_{\boldsymbol{X}}(\boldsymbol{x}, \boldsymbol{y}, t) &= e^{\boldsymbol{\mu}(\boldsymbol{y}-\boldsymbol{x})^T - \left(\boldsymbol{\mu}\ \boldsymbol{\sigma}^2 \boldsymbol{\mu}^T\right)\frac{t}{2}} \\ &\quad \times \int_{(0,\boldsymbol{x})}^{(t,\boldsymbol{y})} D\boldsymbol{x}_\eta(s) e^{-\frac{1}{2}\int_0^t \dot{\boldsymbol{x}}_\eta^2(s) ds - \frac{\boldsymbol{\mu}}{2}\int_0^t \boldsymbol{\eta}(\boldsymbol{x}_\eta(s))^T dL_s(\boldsymbol{x}_\eta)}. \end{aligned} \tag{15}$$

The path integral representation for the two-dimensional η-reflected geometric Brownian motion is obtained using the change of variable relation

$$p_{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{y}, t) = p_{\boldsymbol{X}}(\ln \boldsymbol{x}, \ln \boldsymbol{y}, t) \times |\boldsymbol{J}(\boldsymbol{y})|$$

where $\boldsymbol{J}(\boldsymbol{x})$ is the Jacobian matrix associated to the function $\ln \boldsymbol{x}$. The transition density $p_{\boldsymbol{F}}(\boldsymbol{x}, \boldsymbol{y}, t)$ is hence the solution of the backward Fokker-Planck equation

$$\begin{cases} \frac{d}{dt} I(\boldsymbol{x}, \boldsymbol{y}, t) = \mathcal{G}\ I(\boldsymbol{x}, \boldsymbol{y}, t), & (\boldsymbol{x}, \boldsymbol{y}, t) \in \mathcal{A}^2 \times \mathbb{R}^+ \\ I(\boldsymbol{x}, \boldsymbol{y}, 0) = \delta(\boldsymbol{x} - \boldsymbol{y}), & (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}^2 \\ \boldsymbol{\eta}(\boldsymbol{x}) \cdot \nabla I(\boldsymbol{x}, \boldsymbol{y}, t) = 0, & (\boldsymbol{x}, \boldsymbol{y}, t) \in \partial\mathcal{A} \times \mathcal{A} \times \mathbb{R}^+ \end{cases}$$

where $\mathcal{G}$ is the following operator

$$\mathcal{G} = \frac{1}{2} \sum_{i,j=1}^{2} \sigma_{i,j}^2 x_i x_j \frac{d^2}{dx_i dx_j} + \sum_{i=1}^{2} \left(\mu_i + \frac{1}{2}\sigma_i^2\right) x_i \frac{d}{dx_i}. \tag{16}$$

3 Asset-Liability Management

Provided an ALM policy $(\mathcal{A}, \boldsymbol{\eta}(\boldsymbol{F}))$, the total value creation at the upper barrier is given by the formula

$$C_t^{(2)}(\boldsymbol{F}) = \frac{1}{2} \sum_{i=1}^{2} \int_0^t e^{-rs} |\eta_i^{(\lambda_2)}| dL_s^{\lambda_2,-}(\boldsymbol{F}). \tag{17}$$

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[3] For sake of completness, we have used $\frac{d}{d\boldsymbol{x}} = \left(\frac{d}{dx_1}, \frac{d}{dx_2}\right)$, $\frac{d^2}{d\boldsymbol{x}^2} = \begin{pmatrix} \frac{d^2}{dx_1^2} & \frac{d^2}{dx_1 dx_2} \\ \frac{d^2}{dx_1 dx_2} & \frac{d^2}{dx_2^2} \end{pmatrix}$, $\frac{d}{d\boldsymbol{x}} e^{\boldsymbol{\mu} \cdot \boldsymbol{x}^T} = \boldsymbol{\mu} \cdot e^{\boldsymbol{\mu} \cdot \boldsymbol{x}^T}$, and $\frac{d^2}{d\boldsymbol{x}^2} e^{\boldsymbol{\mu} \cdot \boldsymbol{x}^T} = \boldsymbol{\mu}^T \boldsymbol{\mu} \cdot e^{\boldsymbol{\mu} \cdot \boldsymbol{x}^T}$.

On the other hand, the value annihilation at the lower barrier needed to avoid bankruptcy is defined by a similar formula

$$C_t^{(1)}(\boldsymbol{F}) = \frac{1}{2}\sum_{i=1}^{2}\int_0^t e^{-rs}|\eta_i^{(\lambda_1)}|dL_s^{\lambda_1,+}(\boldsymbol{F}). \tag{18}$$

A perfect ALM strategy consists of continuously matching the amount needed at the lower barrier with the excess at the upper barrier:

$$C_t^{(2)}(\boldsymbol{F}) = C_t^{(1)}(\boldsymbol{F}), \quad \forall t.$$

In what follows, we show how path integrals can be used to express the moment-generating function of the cash flow $C_t^{(i)}(\boldsymbol{F})$ and derive partial differential equations for their expectations. To start with, we model the firm value by means of a Brownian motion with drift. In the second part, we assume that the firm value is driven by a more realistic geometric Brownian motion. Explicit solutions are only provided in case $\boldsymbol{\eta}^{(\lambda_2)} = (-1, 0)$ and $\boldsymbol{\eta}^{(\lambda_1)} = (1, 0)$ for $C_t^{(2)}(\boldsymbol{F})$. We are confident that the interested reader will be able to adapt the results in case $\boldsymbol{\eta}^{(\lambda_2)} = (0, 1)$ and $\boldsymbol{\eta}^{(\lambda_1)} = (0, -1)$ as well as for $C_t^{(1)}(\boldsymbol{F})$

3.1 The case of an ordinary Brownian motion

Assume that the firm value $\boldsymbol{F} = (A, L)$ follows a two-dimensional Brownian motion as described in (1). Consider an ALM policy that consists of maintaining the asset value of the firm in the corridor

$$L + \lambda_1 \leq A \leq L + \lambda_2$$

with $\lambda_1 < \lambda_2$ (λ_1 can be a small negative number). We also assume that the ALM committee acts only if the assets value of the firm lies near the limits of the allowable range. Whenever the assets meet the boundaries, the ALM committee adjusts the assets and the liabilities according to the reflection vector $\boldsymbol{\eta}(\boldsymbol{F})$. Under the ALM policy $(\mathcal{A}, \boldsymbol{\eta}(\boldsymbol{F}))$, the cash flow $C_t^{(i)}(\boldsymbol{F})$ depends only on the surplus $S(t) = A(t) - L(t)$ of the firm.
The quantum description of the firm enables us to formulate the moment-

generating function of $C_t^{(i)}(\boldsymbol{F})$ by means of the following path integral

$$\begin{aligned} \mathrm{E}_{\boldsymbol{x}}\left[e^{\alpha C_t^{(i)}(\boldsymbol{F})}\right] &= \mathrm{E}\left[e^{\alpha C_t^{(i)}(\boldsymbol{F})} | \boldsymbol{F}(0) = \boldsymbol{x}\right] \\ &= \int d\boldsymbol{y}\ e^{\boldsymbol{\mu}(\boldsymbol{y}-\boldsymbol{x})^T - \left(\boldsymbol{\mu}\ \boldsymbol{\sigma}^2 \boldsymbol{\mu}^T\right)\frac{t}{2}} \\ &\quad \times \int_{(0,\boldsymbol{x})}^{(t,\boldsymbol{y})} D\boldsymbol{x}_\eta(s) e^{\alpha C_t^{(i)}(\boldsymbol{x}_\eta)} e^{-\frac{1}{2}\int_0^t \dot{\boldsymbol{x}}_\eta^2(s)ds - \frac{\boldsymbol{\mu}}{2}\int_0^t \boldsymbol{\eta}(\boldsymbol{x}_\eta(s))^T dL_s(\boldsymbol{x}_\eta)}. \end{aligned}$$

In the simple case $\boldsymbol{\eta}^{(\lambda_1)} = (1,0)$ and $\boldsymbol{\eta}^{(\lambda_2)} = (-1,0)$ (the ALM committee only acts on the assets, which at the upper barrier corresponds to a payment of dividends), the problem can be reduced to a one-dimensional path integration. The cash flow at the upper barrier satisfies

$$\begin{aligned} C_t^{(2)}(\boldsymbol{F}) &= \frac{1}{2}\int_0^t e^{-rs} dL_s^{\lambda_2,-}(\boldsymbol{F}) \\ &= \frac{1}{2}\lim_{\epsilon \to 0}\int_0^t ds\ e^{-rs} 1_{(-\epsilon,0)}\left(A(s) - L(s) - \lambda_2\right). \end{aligned}$$

In this particular situation, we can check that the surplus $S(t)$ is a reflected one-dimensional Brownian motion on $[\lambda_1, \lambda_2]$ with drift $\mu_1 - \mu_2$ and volatility $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$:

$$dS(t) = (\mu_1 - \mu_2)dt + \sigma dB(t) + \frac{1}{2}d\left(L_t^{\lambda_1,+}(S) - L_t^{\lambda_2,-}(S)\right) \tag{19}$$

where $B(t)$ is a scalar Brownian motion, $L_t^{\lambda_1,+}(S)$ and $L_t^{\lambda_2,-}(S)$ are respectively the right and left local time of S at the boundaries. A path integral representation for the transition density $p_S(z_0, z, t)$ of S has been derived in Decamps, De Schepper and Goovaerts (2004), see also Carreau (1992) and Grosche (1990, 1993).

We are also interested in the expectation of $C_t^{(2)}(\boldsymbol{F})$

$$\begin{aligned} \mathrm{E}_{z_0}\left[C_t^{(2)}(\boldsymbol{F})\right] &= \mathrm{E}\left[C_t^{(2)}(\boldsymbol{F}) | A(0) - L(0) = z_0\right] \\ &= \frac{1}{2}\mathrm{E}_{z_0}\left[\int_0^t e^{-rs} dL_s^{\lambda_2,-}(S)\right] \\ &= \frac{1}{2}\int_0^t ds\ e^{-rs}\ p_S(z_0, \lambda_2, s). \end{aligned} \tag{20}$$

The steady-state solution $\mathrm{E}_{z_0}\left[C_{+\infty}^{(2)}(\boldsymbol{F})\right]$ is given by $\frac{1}{2}G(z_0, \lambda_2, r)$, where $G(z_0, z, s)$ is the Green function of $p_S(z_0, z, t)$ which can be obtained by

δ-function perturbation. We then retrieve the usual peculiar differential equation for optimal asset-liability management with unbounded cash flows for $\mathrm{E}_z\left[C_t^{(2)}(\boldsymbol{F})\right]$ as

$$\frac{d}{dt}K(z,t) = \left[\frac{1}{2}\sigma^2\frac{d^2}{dz^2} + (\mu_1-\mu_2)\frac{d}{dz} - r\right]K(z,t),$$

for $\lambda_1 \le z = a - l \le \lambda_2$, see Gerber and Shiu (2004) and Taksar (1997, 1999), subject to smooth pasting boundary conditions

$$\frac{d}{dz}K(\lambda_2,t) = 1, \qquad \frac{d}{dz}K(\lambda_1,t) = 0.$$

A partial differential equation for the higher moments can be derived from the path integral representation of the moment-generating function[4].

3.2 The case of a geometric Brownian motion

A two-dimensional geometric Brownian motion offers a more realistic model for the firm value. The value of the firm is now $\boldsymbol{F} = (A, L)$ as described in (6). Consider a ALM policy that consists of maintaining the assets value of the firm in the cone

$$\lambda_1 L \le A \le \lambda_2 L$$

with $0 \le \lambda_1 < \lambda_2$, where λ_1 can be somewhat smaller than 1. Whenever the firm value is near the frontier $\partial\mathcal{A}$, the ALM committee adjusts the assets and the liabilities according to the reflection vector $\boldsymbol{\eta}(\boldsymbol{F})$. Under the ALM policy $(\mathcal{A}, \boldsymbol{\eta}(\boldsymbol{F}))$, the (modified) time-value of the firm is a two-dimensional η-reflected geometric Brownian motion .

In the simple case $\boldsymbol{\eta}^{(\lambda_1)} = (1,0)$ and $\boldsymbol{\eta}^{(\lambda_2)} = (-1,0)$, the ratio $R(t) = A(t)/L(t)$ is a one-dimensional reflected geometric Brownian motion and the cash flow at the upper barrier satisfies

$$\begin{aligned} C_t^{(2)}(\boldsymbol{F}) &= \frac{1}{2}\int_0^t e^{-rs} dL_s^{\lambda_2,-}(\boldsymbol{F}) \\ &= \frac{1}{2}\lim_{\epsilon\to 0}\int_0^t ds\; e^{-rs} 1_{(-\epsilon,0)}\left(A(s) - \lambda_2 L(s)\right) \\ &= \frac{1}{2}\int_0^t e^{-rs} F_1(s)\; dL_s^{\ln\lambda_2,-}(\ln \boldsymbol{F}), \end{aligned}$$

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[4] We just need to insert the series $\mathrm{E}_z\left[e^{\alpha C_t^{(2)}(\boldsymbol{F})}\right] = \sum_{k=0}^{+\infty}\frac{(-1)^k\alpha^k}{k!}\mathrm{E}_z\left[\left(C_t^{(2)}(\boldsymbol{F})\right)^k\right]$ into the two-dimensional Schrödinger equation.

the last identity follows from relation (10). The first moment of the cash flow $C_t^{(2)}(\boldsymbol{F})$ satisfies

$$\begin{aligned} \mathrm{E}_{\boldsymbol{x}}\left[C_t^{(2)}(\boldsymbol{F})\right] &= \mathrm{E}\left[C_t^{(2)}(\boldsymbol{F})|\boldsymbol{F}(0)=\boldsymbol{x}\right] \\ &= \frac{1}{2}\mathrm{E}_{\boldsymbol{x}}\left[\int_0^t e^{-rs} dL_s^{\lambda_2,-}(\boldsymbol{F})\right] \\ &= \frac{1}{2}\int_0^t ds \ \ e^{-rs} \int_{e(\lambda_2)} d\boldsymbol{y} \ \ p_{\boldsymbol{F}}(\boldsymbol{x},\boldsymbol{y},s). \end{aligned}$$

The steady-state solution here is $\frac{1}{2}\int_{e(\lambda_2)} d\boldsymbol{y}\ G(\boldsymbol{x},\boldsymbol{y},r)$, where $G(\boldsymbol{x},\boldsymbol{y},s)$ is the Green function of $p_{\boldsymbol{F}}(\boldsymbol{x},\boldsymbol{y},t)$. The expectation $\mathrm{E}_{\boldsymbol{x}}\left[C_t^{(2)}(\boldsymbol{F})\right]$ can thus be found as the solution of the following two-dimensional partial differential equation

$$\frac{d}{dt}K(\boldsymbol{x},t) = \mathcal{G}\ K(\boldsymbol{x},t), \quad \boldsymbol{x}\in\mathcal{A} \tag{21}$$

where the operator $\mathcal{G}$ is the infinitesimal generator of the two-dimensional geometric Brownian motion $\boldsymbol{F}$ as defined in Section 2. The partial differential equation (21) can be reduced using the scaling property $K(\alpha\boldsymbol{x},t) = \alpha K(\boldsymbol{x},t)$:

$$\begin{aligned} K((x_1,x_2),t) &= x_2 K\left(\left(\frac{x_1}{x_2},1\right),t\right) \\ &= x_2 K\left((z,1),t\right), \end{aligned} \tag{22}$$

the boundary conditions also simplify as $R(t) = F_1(t)/F_2(t)$ is a reflecting diffusion on $[\lambda_1,\lambda_2]$. In view of the identities

$$\begin{aligned} \frac{d}{dx_2}K(\boldsymbol{x},t) &= K((z,1),t) \\ \frac{d^2}{dx_2^2}K(\boldsymbol{x},t) &= 0 \\ \frac{d^2}{dx_1dx_2}K(\boldsymbol{x},t) &= \frac{d}{dz}K((z,1),t), \end{aligned}$$

we finally recover the peculiar equation (5.15) in Gerber and Shiu (2004) (normalizing x_2 to 1)

$$\frac{d}{dt}K(z,t) = \left[\frac{1}{2}\sigma^2 z^2\frac{d^2}{dz^2} + (\delta_1-\delta_2)z\frac{d}{dz} + (\delta_2 - r)\right]K(z,t), \tag{23}$$

for $\lambda_1 \leq z = a/l \leq \lambda_2$ where $\delta_1 = \mu_1 + \frac{1}{2}\sigma_1^2$ and $\delta_2 = \mu_2 + \frac{1}{2}\sigma_2^2$, subject to smooth pasting boundary conditions

$$\frac{d}{dz}K(\lambda_2, t) = 1, \qquad \frac{d}{dz}K(\lambda_1, t) = 0.$$

4 Path integral solutions to the peculiar equations in the case of a geometric Brownian motion

Gerber and Shiu (2004) provide closed-form expressions for the discounted cash flows at infinite time-horizon. As pointed out by Gerber and Shiu (2004), the boundness of the average cash flow is guaranteed by the condition $r > \delta_2$, a discount factor greater than the growth of liabilities. The steady-state solution $K(z) = K(z, +\infty)$ satisfies the equation

$$\left[\frac{1}{2}\sigma^2 z^2 \frac{d^2}{dz^2} + (\delta_1 - \delta_2)z\frac{d}{dz} + (\delta_2 - r)\right] K(z,t) = 0, \quad z = a/l,$$

which is nothing else but the Euler ordinary differential equation. We look for solution in the form $K(z) = C_1 z^{\theta_1} + C_2 z^{\theta_2}$ and we obtain that

$$K(z) = \lambda_1 \frac{h(z/\lambda_1)}{h'(\lambda_2/\lambda_1)} \tag{24}$$

where $h(x) = \frac{x^{\theta_1}}{\theta_1} - \frac{x^{\theta_2}}{\theta_2}$, θ_1 and θ_2 are the zeros of the indicial equation

$$\frac{\sigma^2}{2}\theta^2 + \left(\delta_1 - \delta_2 - \frac{\sigma^2}{2}\right)\theta + \delta_2 - r = 0.$$

We refer to Gerber and Shiu (2004) for indepth study of this equation[5]. For finite time-horizon, we rely on path integrals to compute $K(z,t)$. Assume that we start at time $t = 0$ with the initial endowment $K(z)$ needed to fund the payment of dividends at the upper boundary $e(\lambda_2)$ up to infinity.

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[5] Similarly, Gerber and Shiu (2004) find the expression for the average discounted cash flow at the lower barrier:

$$K(z) = -\lambda_2 \frac{h(z/\lambda_2)}{h'(\lambda_1/\lambda_2)}.$$

Whenever the asset value of the firm is in the neighborhood of the edge $e(\lambda_2)$, this endowment is reduced by an amount proportional to the local time on $e(\lambda_2)$. Let $J(z,t)$ be the remaining amount of money at time t:

$$J(z,t) = K(z) - K(z,t).$$

The surplus $J(z,t)$ is a positive solution of the partial differential equation (23) subject to the conditions

$$J(z,0) = K(z), \quad \frac{d}{dz}J(\lambda_1,t) = 0, \quad \frac{d}{dz}J(\lambda_2,t) = 0.$$

The surplus $J(z,t)$ is hence the solution of the Fokker-Planck equation (23) subject to reflected boundary conditions and is obtained by convolution of the initial condition with the kernel solution of (23):

$$J(z,t) = \int_{\lambda_1}^{\lambda_2} K(z_0) e^{(\delta_2 - r)t} e^{\overline{\mu}(z_0 - z) - \overline{\mu}^2 \frac{t}{2}} I(\phi(z), \phi(z_0), t) \frac{dz_0}{z_0},$$

with $\phi(z) = \ln z/\sigma$ and with $\overline{\mu} = \delta_1 - \delta_2 - \frac{1}{2}\sigma^2$. The function $I(z_0,z,t)$ can be expressed by means of the following (one-dimensional) path integral over reflected Brownian paths

$$I(z_0,z,t) = \int_{(0,z_0)}^{(t,z)} D|z(s)| e^{-\frac{1}{2}\int_0^t |\dot{z}(s)|^2 ds - \frac{\overline{\mu}}{2}\left(L_t^{\phi(\lambda_1^+)}(|z|) - L_t^{\phi(\lambda_2^-)}(|z|)\right)}, \tag{25}$$

see Decamps, De Schepper and Goovaerts (2004). In what follows, we provide closed-form expressions for the path integral $I(z_0,z,t)$ in case $\lambda_1 = 0$ and $\lambda_1 \neq 0$.

4.1 In absence of a lower barrier ($\lambda_1 = 0$)

In the simple case $\lambda_1 = 0$, the path integral $I(z_0,z,t)$ is the transition density of a Brownian motion on $(-\infty, \phi(\lambda_2)]$ reflected at $\phi(\lambda_2)$ and killed when the time spent in the vicinity of $\phi(\lambda_2)$ (measured by the local time) exceeds an independent exponential random variable with parameter $\overline{\mu}$. This process called elastic Brownian motion was first mentioned by Itô and McKean (1974). This path integral is computed in Decamps, De Schepper and Goovaerts (2004) using δ-function perturbation. For sake of completeness, we recall the expression obtained for $I(z_0,z,t)$:

$$\begin{aligned} I(z_0,z,t) &= \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{1}{2t}(z-z_0)^2} + e^{-\frac{1}{2t}(z+z_0-2\phi(\lambda_2))^2}\right) \\ &\quad -\overline{\mu} e^{\overline{\mu}k} e^{\frac{1}{2}\overline{\mu}^2 t} \operatorname{erfc}\left(\overline{\mu}\sqrt{t/2} + \frac{k}{\sqrt{2t}}\right) \end{aligned}$$

with $k = (z + z_0 - 2\phi(\lambda_2))$ and $z_0, z \leq \phi(\lambda_2)$.

4.2 In presence of a lower barrier ($\lambda_1 > 0$)

In case $\lambda_1 > 0$, the path integral $I(z_0, z, t)$ is the transition density of a Brownian motion on $[\phi(\lambda_1), \phi(\lambda_2)]$ elastically reflected at $\phi(\lambda_1)$ and $\phi(\lambda_2)$. There exists no closed-form expression for $I(z_0, z, t)$; however, we can obtain its spectral decomposition. The path integral $I(z_0, z, t)$ can be decomposed as the sum

$$I(z_0, z, t) = \sum_E e^{-Et} \; \Psi_E(z_0)\Psi_E(z) \tag{26}$$

where $\Psi_E(z)$ is the properly normalized eigenfunction associated to the level of energy E. As the domain $[\phi(\lambda_1), \phi(\lambda_2)]$ is bounded when $\lambda_1 > 0$, the spectrum is a discrete sequence $\{E_i\}_{i \in \mathbb{N}}$ and the spectral decomposition (26) reduces to a series expansion. The eigenfunction $\Psi_E(z)$ satisfies the Sturm-Liouville equation of the free particle, $-\frac{1}{2}\frac{d^2}{dz^2}\Psi(z) = E\Psi(z)$, subject to the following boundary conditions

$$\frac{d}{dz}\Psi(z) = \overline{\mu} \; \Psi(z)|_{\phi(\lambda_1)}, \quad \frac{d}{dz}\Psi(z) = \overline{\mu} \; \Psi(z)|_{\phi(\lambda_2)}, \tag{27}$$

see Decamps, De Schepper and Goovaerts (2004). We look for eigenfunctions in the form

$$\Psi_E(z) = \cos\left(\sqrt{2E}z\right) + c \; \sin\left(\sqrt{2E}z\right).$$

Inserting $\Psi_E(z)$ in the conditions (27), we obtain that

$$c = \frac{\overline{\mu}\cos(\sqrt{2E}\ln\lambda_1/\sigma) + \sqrt{2E}\sin(\sqrt{2E}\ln\lambda_1/\sigma)}{-\overline{\mu}\sin(\sqrt{2E}\ln\lambda_1/\sigma) + \sqrt{2E}\cos(\sqrt{2E}\ln\lambda_1/\sigma)}$$

and the eigenvalues E_i can be found numerically as the solutions of the following equation

$$\left(c\sqrt{2E} + \overline{\mu}\right)\cos(\sqrt{2E}\ln\lambda_2/\sigma) = \left(\sqrt{2E} + \overline{\mu}\; c\right)\sin(\sqrt{2E}\ln\lambda_2/\sigma).$$

5 Numerical illustration

In this section, we illustrate numerically the convergence of the average discounted cash flow $K(z, t)$ to the stationary solution $K(z)$. Assume the firm value $\boldsymbol{F} = (A, L)$ follows a two-dimensional geometric Brownian motion with drift $\boldsymbol{\mu} = (0.6, 0.4)$, $\sigma_1 = 0.2$, $\sigma_2 = 0.15$ and $\rho = 0.2$. The ALM policy is determined by $(\mathcal{A}, \boldsymbol{\eta}(\boldsymbol{F}))$ where $\mathcal{A}$ is the allowable set $\lambda_1 L \leq A \leq \lambda_2 L$ with $\lambda_1 = 0.7$ and $\lambda_2 = 1.2$, $\boldsymbol{\eta}^{(\lambda_1)} = (1, 0)$ and $\boldsymbol{\eta}^{(\lambda_2)} = (-1, 0)$. Figure 1 presents the stationary solutions for the cash flows at the upper and lower barriers. For this particular choice of parameters, it exists a value in the interval $[\lambda_1, \lambda_2]$ such that the average discounted overflow at the upper barrier is equal to the average discounted cash flows at the lower barrrier needed to maintain the firm value in the allowable set $\mathcal{A}$. Gerber and Shiu (2004) calls this particular value of the ratio $z = a/l$ the *Cost-Neutral* initial value. A sustainable ALM policy should at least ensure the existence of this solution. Figure 2 illustrates the convergence of $K(z, t)$ to the stationary solution for the cost-neutral initial value. The solutions converge only after 35 years. It emphasizes the relevance of both a long and short term analysis of the ALM strategy. In this case, the dividend paid out at the upper barrier remains above the fresh capital needed at the lower barrier, which reveals a better transformation of value in the start-up phase of the business cycle.

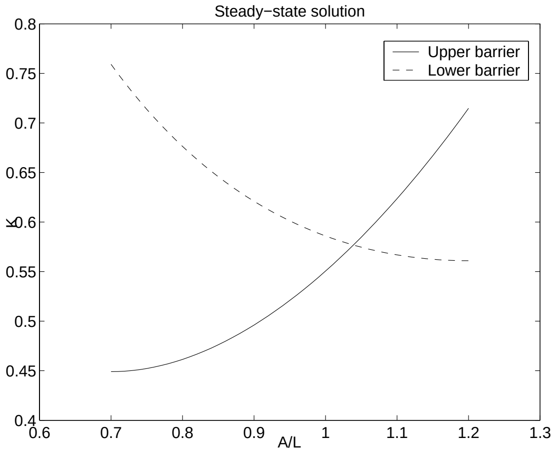

Figure 1: Average cash flows at the lower and upper barriers.

6 Conclusion

In this paper, we showed how path integrals can generate a powerful tool for the study of cash flows generated by a given ALM policy $(\mathcal{A}, \boldsymbol{\eta}(\boldsymbol{F}))$.

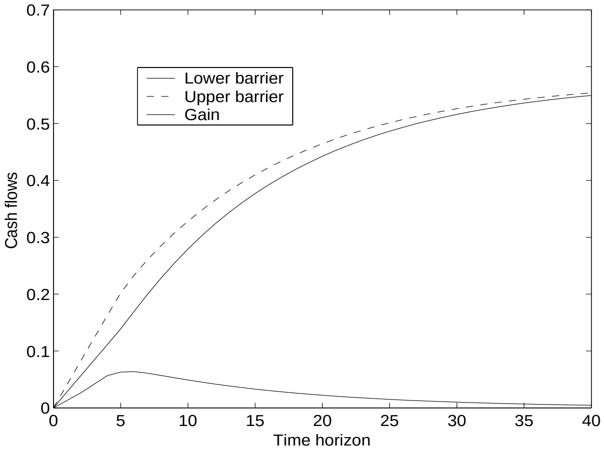

Figure 2: Convergence of the average discounted cash flows at the lower and upper barriers.

Although it might be complicated, this approach can provide new insight into ALM. It will lead to numerical schemes, even for more realistic processes for the firm value, e.g. non-Gaussian to include jumps. Finally, the path integral formulation can also be used to calculate the probability to exit of the allowable cone $\mathcal{A}$, which appear to be useful for credit risk management as well as for the pricing of credit derivatives as e.g. credit default swap.

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