Asymmetric skew Bessel processes and their applications to finance

Marc Decamps[*], Marc Goovaerts[†], Wim Schoutens[‡]

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Abstract

In this paper, we extend the Harrison and Shepp's construction of the skew Brownian motion (1981) and we obtain a diffusion similar to the two-dimensional Bessel process with speed and scale densities discontinuous at one point. Natural generalizations to multi-dimensional and fractional order Bessel processes are then discussed as well as invariance properties. We call this family of diffusions *asymmetric* skew Bessel processes in opposition to skew Bessel processes as defined in Barlow, Pitman and Yor (1989). We present factorizations involving (asymmetric skew) Bessel processes with random time. Finally, applications to the valuation of perpetuities and Asian options are proposed.

Keywords: Bessel processes, local time, perpetuities, Asian options

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[*] Marc Decamps is assistant at the K.U.Leuven, FETEW, Naamsestraat 69, B-3000 Leuven, e-mail: marc.decamps@econ.kuleuven.ac.be.

[†] Marc Goovaerts is full Professor at the K.U.Leuven and at the U.v.Amsterdam, FETEW, Naamsestraat 69, B-3000 Leuven, e-mail: marc.goovaerts@econ.kuleuven.ac.be.

[‡] Wim Schoutens is postdoctoral researcher at the K.U.Leuven, Department of Mathematics, Celestijnenlaan 200B, B-3001 Leuven, e-mail: Wim.Schoutens@wis.kuleuven.ac.be.

1 Introduction

The *skew Brownian motion* (skew BM) was first mentioned by Itô and McKean (1974). Since then many authors have been interested in this diffusion process . We cite Walsh (1978), Harrison and Shepp (1981), Le Gall (1982) and Ouknine (1991). A skew Brownian motion with parameter $0 \leq \beta \leq 1$ behaves like a Brownian motion away from the origin and is reflected to the positive side with probability β and to the negative side with probability $1-\beta$ when it hits 0. As shown in detail by Walsh (1978), the resulting process is a linear diffusion with discontinuous scale and speed densities. Harrison and Shepp (1981) construct the skew Brownian motion from a piecewise linear function of a time changed Brownian motion and prove, using Tanaka formula, that it is a unique strong solution to the Stochastic Differential Equation (SDE)

$$dX_\beta(t) = (2\beta - 1)dL_t^0(X_\beta) + dB(t) \tag{1}$$

where $B = \{B(t), t \geq 0\}$ is an adapted Brownian motion and $L_t^0(X_\beta)$ is the symmetric local time of the continuous semimartingale X_β at 0. The transition density of the skew BM has a discontinuous derivative and is obtained via its Green function. The intriguing properties of the skew Brownian motion have led to applications in various disciplines. We can cite Zhang (2000) in theoretical physics or Cantrell and Cosner (1999) in biology.

A Bessel process of order ν, denoted by $BES^{(\nu)}$, is a linear diffusion with generator

$$\mathcal{G}^{(\nu)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}. \tag{2}$$

If one multiplies by x^2 the *fundamental equation* $\mathcal{G}^{(\nu)}u = \alpha u$, we recover the modified Bessel's differential equation. When $\nu = -1/2, 0, 1/2, 1, \ldots$ the $BES^{(\nu)}$ can be represented as the distance from the origin of a $d = (2\nu+2)$-dimensional Brownian motion. Bessel processes and some generalizations have been investigated for a long time in financial mathematics. It plays an essential role for evaluating Asian option and contingent claims under the CIR model, see Yor (2001). For an extensive survey on Bessel processes, we refer to Going-Jaeschke and Yor (2003). From a theoretical point of view, Bessel processes together with the Brownian motion are often considered as reference processes for their numerous properties. In particular the duality between the three-dimensional Bessel process and the Brownian motion killed at the origin guides to surprising results as for instance the Williams' path decomposition and the Pitman's theorem, see *e.g.* Yor (1995) and Revuz and Yor (1998).

Barlow, Pitman and Yor (1989) construct the skew Bessel process by changing the sign with probability $1-\beta$ of the excursions away from 0 of a Bessel process of dimension $d \in (0,2)$. The symmetry of the skew Bessel processes introduced by Barlow et al.(1989) has lead to interesting developments, see *e.g.* Watanabe (1995) and (1998)[1]. In this paper, we propose an asymmetric definition of two dimensional skew Bessel process by introducing a discontinuity at some level $a \in (0,+\infty)$ of the scale and speed densities. Although the symmetry property no longer holds, we show that this family of processes shares relevant properties with the skew BM and the Bessel processes. It also provides some insight in the understanding of the exponential functionals

$$A(t) = \int_0^t e^{2X_\beta(s)} ds$$

and

$$\hat{A}(t) = \beta^2 \int_0^t e^{2B(s)} 1_{B(s)<0} ds + (1-\beta)^2 \int_0^t e^{2B(s)} 1_{B(s)\geq 0} ds.$$

The rest of the paper is organized as follows. We start to define the asymmetric skew Bessel process of dimension two from its discontinuous speed and scale densities. We write for short $R_\beta^{(2)} = \{R_\beta^{(2)}(t), t \geq 0\}$ is a BES_β^2. Standard theorems on linear diffusions permit to derive the infinitesimal generator and its domain. Parellel to the Harrison and Shepp's construction of the skew BM, we show that $R_\beta^{(2)}$ is a continuous semimartingale solution of a stochastic differential equation involving its local time. In a second part, we generalize our definition to fractional orders and we briefly discuss invariance properties. We resurrect the generalized diffusions introduced by Portenko (1979) to recover the radial property of Bessel processes of dimension $d = 2, 3, \ldots$. We then present Lamperti like relations involving (asymmetric skew) Bessel processes with random time. Finally, applications to the valuation of perpetuities and Asian options are proposed.

2 Construction

We construct the BES_β^2 from its scale function and speed measure. Standard theorems on linear diffusions permit to derive the infinitesimal genera-

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[1] We are endebted to Marc Yor for bringing to our attention those recent papers.

tor and its domain. We briefly discuss how to compute the Green function. Similarly to Harrison and Shepp (1981), we prove that the resulting process is solution of a stochastic differential equation involving its local time.

2.1 Definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 1 *A Bessel process BES^2_β skewed by $\beta \in (0,1)$ at $a > 0$ is a linear diffusion with speed measure $m(dx) = m(x)dx$ where*

$$m(x) = \begin{cases} \frac{2x}{(1-\beta)}, & x \geq a \\ \frac{2x}{\beta}, & x < a \end{cases} \tag{3}$$

and scale function $s(x) = \int^x s'(y)dy$ where

$$s'(x) = \begin{cases} \frac{(1-\beta)}{x}, & x \geq a \\ \frac{\beta}{x}, & x < a. \end{cases} \tag{4}$$

Remark 1 *This diffusion takes values on the interval $I = [0, +\infty)$. As $\int_0^{1/2} m((y, 1/2])s(dy) = \infty$ but $\int_0^{1/2} m((0, y])s(dy) < \infty$, $+\infty$ is the right natural boundary, 0 is an entrance-not-exit boundary. Thus a BES^2_β can be started in 0 but will never reach 0 later and cannot reach $+\infty$ within a finite time.*

The infinitesimal generator results from a direct application of Theorem VII.3.12. in Revuz and Yor (1998, p308) and differs only from the generator of the BES^2 to the restriction on the domain.

Lemma 1 *The infinitesimal generator of the BES^2_β reads*

$$\mathcal{G}f = \frac{1}{2}\frac{d^2 f}{dx^2} + \frac{1}{2x}\frac{df}{dx}, \quad x > 0, x \neq a$$

and $\mathcal{G}f(a) = \frac{1}{2}f''(a+) + \frac{1}{2a}f'(a+) = \frac{1}{2}f''(a-) + \frac{1}{2a}f'(a-)$ acting on the domain

$$\mathcal{D} = \left\{ f : f, \mathcal{G}f \in C_b([0,\infty)), \frac{df^+}{ds}(0+) = 0, (1-\beta)f'(a-) = \beta f'(a+) \right\}$$

where $\frac{df}{ds}$ is the s-derivative defined by $\frac{df^+}{ds}(x) = lim_{h \to 0+} \frac{f(x+h) - f(x)}{s(x+h) - s(x)}$ and $\frac{df^-}{ds}(x) = lim_{h \to 0+} \frac{f(x) - f(x-h)}{s(x) - s(x-h)}$.

Proof. The definition of $\mathcal{G}$ on $(0,\infty)\setminus\{a\}$ follows from Theorem $VII.3.12.$ in Revuz and Yor (1998),

$$\begin{aligned}\int_{[b,c)}\left(\frac{1}{2}f''(x)+\frac{1}{2x}f(x)\right)m(dx) &= \frac{m(x)}{2}f'(x)|_b^c\\ &= \frac{df^-}{ds}(c)-\frac{df^-}{ds}(b),\\ &= \frac{df^+}{ds}(c)-\frac{df^+}{ds}(b).\end{aligned}$$

$\mathcal{G}f(a)=\frac{1}{2}f''(a+)+\frac{1}{2a}f'(a+)=\frac{1}{2}f''(a-)+\frac{1}{2a}f'(a-)$ comes from the condition that $\mathcal{G}f\in C_b(I)$. As $m(dx)$ is absolutely continuous w.r.t. the Lebesgue measure, $\int_{[a,b)}\mathcal{G}f(x)m(dx)=\int_{(a,b]}\mathcal{G}f(x)m(dx)$, so $\frac{df^+}{ds}(a)=\frac{df^-}{ds}(a)$ or equivalently $(1-\beta)f'(a-)=\beta f'(a+)$. As 0 is an entrance-not-exit boundary, we obtain the condition $\frac{df^+}{ds}(0+)=0$. □

The transition density *w.r.t* the speed measure solves the backward Fokker-Planck equation

$$\frac{\partial p}{\partial t}(t;x,.)=\mathcal{G}p(t;x,.) \tag{5}$$

subject to the condition $(1-\beta)\frac{dp}{dx}(t;a-,y)=\beta\frac{dp}{dx}(t;a+,y)$. The associated Green function $G_\alpha(x,y)=\int_0^{+\infty}e^{-\alpha t}p(t;x,y)dt$ can be expressed by means of the *fundamental solutions* of $\mathcal{G}u=\alpha u$ and the *Wronskian*[2]. The fundamental solutions of $\mathcal{G}u=\alpha u$, denoted $\psi_\alpha(x)$ and $\varphi_\alpha(x)$, are continuous, monotone and linearly independent functions that satisfy

$$\alpha\int_{[b,c)}u(x)m(dx)=\frac{du^-}{ds}(c)-\frac{du^-}{ds}(b)$$

for all b and c in the interior of I. It follows that the increasing solution $\psi_\alpha(x)$ has the following properties, see Section 4.6. in Itô and McKean (1974):

$$\psi_\alpha(0+)>0,\quad \frac{d\psi_\alpha^+}{ds}(0+)=0,\quad \psi_\alpha(+\infty)=+\infty,\quad \frac{d\psi_\alpha^-}{ds}(+\infty)=+\infty$$

and $\frac{d\psi_\alpha^+}{ds}(a)=\frac{d\psi_\alpha^-}{ds}(a)$. We can construct $\psi_\alpha(x)$ as a linear combination of the modified Bessel functions $I_0\left(x\sqrt{2\alpha}\right)$ and $K_0\left(x\sqrt{2\alpha}\right)$ that span the

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[2] When $\beta\neq 0$ or 1, the inversion of the Laplace transform is tedious as the spectrum of the operator $\mathcal{G}$ is continuous

solutions of $\mathcal{G}u = \alpha u$. With similar conditions, we can find the decreasing solution $\varphi_\alpha(x)$. The Wronskian is defined as

$$w_\alpha := \frac{d\psi_\alpha^+}{ds}(y)\varphi_\alpha(y) - \frac{d\varphi_\alpha^+}{ds}(y)\psi_\alpha(y)$$

and is independent of y. The Green function $G_\alpha(x, y)$ results from

$$G_\alpha(x,y) = \begin{cases} w_\alpha^{-1}\psi_\alpha(x)\varphi_\alpha(y), & x < y \\ w_\alpha^{-1}\psi_\alpha(y)\varphi_\alpha(x), & y \leq x. \end{cases}$$

2.2 Two dimensional skew bessel processes as semimartingales

In this section, we make use of the notion of local time associated to a continuous semimartingale X. We denote by $L_t^{a\pm}(X)$ the right and left local time and by $L_t^a(X) = \left(L_t^{a-}(X) + L_t^{a+}(X)\right)/2$ the symmetric local time. If $L_t^{a+}(X) = L_t^{a-}(X)$, the local time is said to be continuous in a. In the sequel, we will extensively use the invariance under continuous random time changes $\tau(t)$ of local times, $L_{\tau(t)}^a(X) = L_t^a(X(\tau))$, see Exercise (1.27) in Revuz and Yor (1998, p235).

Harrison and Shepp (1981) construct the skew BM as a piecewise linear function of a BM with a stochastic clock. Let $\{B(t), t \geq 0\}$ be a Brownian motion adapted to the natural filtration and $\sigma_\beta(x)$ the function defined by

$$\sigma_\beta(x) = \begin{cases} 1-\beta, & x \geq 0 \\ \beta, & x < 0. \end{cases} \tag{6}$$

The computation of the speed and scale functions yields that the process $X_\beta(t) = r_\beta^{-1}(B(\tau_\beta(t)))$ where $\tau_\beta(t) = \inf\left\{s | \int_0^s du/\sigma_\beta^2(B(u)) > t\right\}$ and $r_\beta(x) = x\sigma_\beta(x)$ is the skew BM. Applying Tanaka formula, we conclude that the skew BM is a solution of the following SDE

$$dX_\beta(t) = (2\beta - 1)dL_t^0(X_\beta) + dB(t),$$

where $L_t^0(X_\beta)$ is the symmetric local time of X_β. The interpretation of the skew BM as a process similar to the BM but reflected with probability β when it hits 0 gives rise to the next Lemma in terms of the local time at the origin.

Lemma 2 *The local times of X_β satisfy*

$$\begin{aligned} L_t^0(X_\beta) &= L_t^0(B) \\ L_t^{0+}(X_\beta) &= 2\beta L_t^0(B) \\ L_t^{0-}(X_\beta) &= 2(1-\beta)L_t^0(B) \end{aligned} \tag{7}$$

where $L_t^0(B)$ is the Brownian local time. As by-product result, the local time of $B(\tau_\beta(t))$ at the origin is continuous and can be expressed by means of $L_t^0(B)$

$$L_t^0(B(\tau_\beta)) = 2(1-\beta)\beta L_t^0(B).$$

Proof. An application of Tanaka formula to the process $Y = |X_\beta|$ yields

$$Y(t) = \beta(t) + L_t^0(X_\beta)$$

with $\beta(t) = \int_0^t sign(X_\beta(s))dB(s)$. The Lévy's characterization Theorem, see Revuz and Yor (1998, p150), states that $\beta(t)$ is a Brownian motion. As the process Y is a positive reflecting BM, we conclude that $L_t^0(X_\beta) = L_t^0(|B|) = L_t^0(B)$ by identification with the Tanaka formula

$$|B(t)| = \int_0^t sign(B(s))dB(s) + L_t^0(B).$$

Following Ouknine (1991), the local time of the skew BM is continuous except in $a = 0$ where $L_t^{0+}(X_\beta) = 2\beta L_t^0(X_\beta)$ and $L_t^{0-}(X_\beta) = 2(1-\beta)L_t^0(X_\beta)$. The second part is a consequence of the relations

$$L_t^a(X_\beta) = \begin{cases} \frac{1}{2\beta(1-\beta)} L_t^0(B(\tau_\beta)), & a = 0 \\ \frac{1}{1-\beta} L_t^{a/(1-\beta)}(B(\tau_\beta)), & a > 0 \\ \frac{1}{\beta} L_t^{a/\beta}(B(\tau_\beta)), & a < 0. \end{cases} \tag{8}$$

□

It is elementary to adapt the construction of Harrison and Shepp (1981) to prove that the BES_β^2 is a continuous semimartingale. We can verify that $R_\beta^{(2)}(t) = s^{-1}(B(T(t)))$ is a BES_β^2 for some continuous time change T and apply Tanaka formula. In order to interpret the BES_β^2, we start from a process similar to the BES^2 rather than from a BM. We set $a = 1$ to avoid tedious notations. Let $Z = \{Z(t), t \geq 0\}$ be the solution of

$$dZ(t) = \frac{b(Z(t))}{2}dt + dB(t)$$

with

$$b(z)=\left\{\begin{array}{ll}\frac{1}{z-\beta}, & z\geq 1\\ \frac{1}{z-(1-\beta)}, & z<1.\end{array}\right. \tag{9}$$

The stochastic process $Y_\beta=\{Y_\beta(t)=Z(\tau_\beta(t)),t\geq 0\}$ where the stochastic clock τ_β satisfies $\tau_\beta(t)=\inf\left\{s|\int_0^s du/\sigma_\beta^2(Z(u))>t\right\}$ with $\sigma_\beta(z)=(1-\beta)1_{z\geq 1}+\beta 1_{z<1}$ is the semimartingale solution of the following SDE

$$\begin{aligned}Y_\beta(t)-Y_\beta(0) &= \int_0^{\tau_\beta(t)}\frac{b\left(Z(s)\right)}{2}ds+B(\tau_\beta(t))\\ &= \int_0^t\sigma_\beta^2(Y_\beta(s))\frac{b\left(Y_\beta(s)\right)}{2}ds+\int_0^t\sigma_\beta(Y_\beta(s))dB(s). \quad (10)\end{aligned}$$

The process Y_β is a diffusion with scale and speed functions defined by

$$\begin{aligned}m_\beta^Y(y) &= \frac{2e^{\int^y b(z)dz}}{\sigma_\beta^2(y)}\\ s_\beta'^Y(y) &= e^{-\int^y b(z)dz}. \quad (11)\end{aligned}$$

The relations $s_X(x)=s_{f(X)}(f(x))$ and $m_X(x)=m_{f(X)}(f(x))f'(x)$ provide the speed and scale densities of the process $r_\beta^{-1}(Y_\beta)$ where $r_\beta(.)$ is the following continuous function

$$r_\beta(x)=\left\{\begin{array}{ll}(1-\beta)x+\beta, & x\geq 1\\ \beta x+(1-\beta), & x<1.\end{array}\right. \tag{12}$$

We can verify that $r_\beta^{-1}(Y_\beta)$ is a BES_β^2. The Tanaka formula states that $r_\beta^{-1}(Y_\beta)$ is a semimartingale and provides the decomposition given in the next Lemma.

Lemma 3 *The BES_β^2 satisfies the following stochastic differential equation*

$$dR_\beta^{(2)}(t)=\frac{1}{2R_\beta^{(2)}(t)}dt+dB(t)+(2\beta-1)dL_t^1(R_\beta^{(2)})$$

where $L_t^1(R_\beta^{(2)})$ is the symmetric local time. $L_t^a(R_\beta^{(2)})$ is continuous except in $a=1$ where $\beta L_t^{1-}(R_\beta^{(2)})=(1-\beta)L_t^{1+}(R_\beta^{(2)})$.

Proof. By standard use of Tanaka formula, we obtain

$$\begin{aligned} dR_\beta^{(2)}(t) &= dr_\beta^{-1}(Y_\beta) \\ &= \frac{1}{\sigma_\beta(Y_\beta(t))}dY(t) + \frac{(2\beta-1)}{\beta(\beta-1)}dL_t^1(Y_\beta) \\ &= \frac{1}{2R_\beta^{(2)}(t)}dt + dB(t) + \frac{(2\beta-1)}{\beta(\beta-1)}dL_t^1(Y_\beta) \qquad (13) \end{aligned}$$

where $L_t^1(Y_\beta)$ is the symmetric local time of Y_β. From the relations

$$\begin{aligned} L_t^{1+}(Y_\beta) &= (1-\beta)L_t^{1+}(R_\beta^{(2)}) \\ L_t^{1-}(Y_\beta) &= \beta L_t^{1-}(R_\beta^{(2)}) \\ L_t^1(R_\beta^{(2)}) &= \left(L_t^{1+}(R_\beta^{(2)}) + L_t^{1-}(R_\beta^{(2)})\right)/2 \end{aligned}$$

and the continuity of $L_t^1(Y_\beta)$, we obtain that

$$dR_\beta^{(2)}(t) = \frac{1}{2R_\beta^{(2)}(t)}dt + dB(t) + (2\beta-1)dL_t^1(R_\beta^{(2)}). \qquad (14)$$

As suggested by Harrison and Shepp (1981), we can apply the Nakao Theorem (1972) to the process Y_β to prove that $R_\beta^{(2)}$ is the pathwise unique solution. □

According to us, the following relation establishes an interesting connection between BES_β^2, Bessel process of dimension 2 and skew BM. The Harrison and Shepp's construction of the skew BM means that $r_\beta(X_\beta(t)) = B\left(\int_0^t r_\beta'^2(X_\beta(s))ds\right)$ where B is a BM and $r_\beta(x) = x\sigma_\beta(x)$. A similar identity holds for BES_β^2 and is very exclusive.

Lemma 4 *Let $R_\beta^{(2)}$ be a BES_β^2 and $f_\beta(.)$ be the continuous function defined by*

$$f_\beta(x) = \begin{cases} x^{1-\beta}, & x \geq 1 \\ x^\beta, & x < 1, \end{cases}$$

then

$$f_\beta\left(R_\beta^{(2)}(t)\right) = R^{(2)}\left(\int_0^t f_\beta'^2\left(R_\beta^{(2)}(s)\right)ds\right)$$

where $R^{(2)}$ is a Bessel process of dimension $\delta = 2$.

Proof. An application of Tanaka formula provides that the process $Y(t) = f_\beta(R_\beta^{(2)}(t))$ satisfies

$$dY(t) = \frac{f'_\beta(R_\beta^{(2)}(t))}{2R_\beta^{(2)}(t)}dt + \frac{1}{2}f''_\beta(R_\beta^{(2)}(t))dt + f'_\beta(R_\beta^{(2)}(t))dB(t)$$

where $f'_\beta(x) = (1-\beta)x^{-\beta}1_{x\geq 1}+\beta x^{\beta-1}1_{x<1}$ and $f''_\beta(x) = -\beta(1-\beta)x^{-\beta-1}1_{x\geq 1}-\beta(1-\beta)x^{\beta-2}1_{x<1}$. It is easy to check that

$$\frac{f'_\beta(x)}{2x} + \frac{1}{2}f''_\beta(x) = \frac{f'^2_\beta(x)}{2f_\beta(x)}$$

and, thus

$$dY(t) = \frac{f'^2_\beta(R_\beta^{(2)}(t))}{2f_\beta(R_\beta^{(2)}(t))}dt + f'_\beta(R_\beta^{(2)}(t))dB(t).$$

Implementing the stochastic time $\tau(t) = \inf\{s| \int_0^s f'^2_\beta(R_\beta^{(2)}(u))du > t\}$, we obtain

$$\begin{aligned} Y(\tau(t)) - Y(0) &= \int_0^{\tau(t)} \frac{f'^2_\beta(R_\beta^{(2)}(s))}{2f_\beta(R_\beta^{(2)}(s))}ds + \int_0^{\tau(t)} f'_\beta(R_\beta^{(2)}(s))dB(s) \\ &= \int_0^t \frac{1}{2Y(\tau(s))}ds + \beta(t) \end{aligned}$$

where $\beta(t)$ is a Brownian motion according to the Dambis, Dubins-Schwarz's Theorem, see Revuz and Yor (1998, p181). □

We will prove in the sequel that the Lamperti factorization holds between the skew BM and the asymmetric Bessel process of dimension 2. We also represent the BES_β^2 as the radial part of some generalized diffusion. Together with Lemma 4, it motivates the definition of the asymmetric Bessel process of dimension 2. For financial applications, it is also convenient to introduce the following processes. Le Gall (1982) considers the following stochastic differential equation:

$$X(t) = X(0) + \int_0^t f(X(s))dB(s) + \int_{\mathbb{R}} \eta(dx)L_t^x(X). \tag{15}$$

If we set $f(x) := 1$ and $\eta(dx) = (2\beta - 1)\delta_{(x-a)} + \frac{(2\nu+1)dx}{2x}$ where $\delta_{(x-a)}$ is the Dirac measure with support $\{a\}$, it permits us to generalize the BES_β^2 to fractional order $\nu \geq 0$.

Definition 2 *An asymmetric skew Bessel process of order $\nu \geq 0$ with skew parameter $\beta \in (0,1)$ and $a > 0$ is a solution of the stochastic differential equation*

$$dR^{\nu}_{\beta}(t) = \frac{2\nu + 1}{2R^{\nu}_{\beta}(t)}dt + dB(t) + (2\beta - 1)dL^{a}_{t}(R^{\nu}_{\beta}), \quad t > 0 \tag{16}$$

and $R^{\nu}_{\beta}(0) = x > 0$. We write for short that the process $R^{\nu}_{\beta} = \{R^{\nu}_{\beta}(t), t \geq 0\}$ is a $BES^{(\nu)}_{\beta}$ or equivalently $R^{(\delta)}_{\beta} = \{R^{(\delta)}_{\beta}(t), t \geq 0\}$ is a BES^{δ}_{β} with dimension $\delta = 2\nu + 2$.

The $BES^{(\nu)}_{\beta}$ is a linear diffusion with discontinuous speed and scale densities and behaves like a $BES^{(\nu)}$ away from the point a. The origin can not be reached from other positive points for $\nu \geq 0$. For $\nu < 0$, equation (16) is not sufficient to define the $BES^{(\nu)}_{\beta}$ as a condition is involved at the origin. We rely on standard theory about Bessel processes of fractional order for the boundary classification. If $\nu \in (-1, 0)$, 0 is a reflecting boundary and the origin is a trap for $\nu \leq -1$. In the next section, representation results on $BES^{(\nu)}_{\beta}$ will involve the skew Brownian motion with drift μ. We write for short $X^{\mu}_{\beta} = \{X^{\mu}_{\beta}(t), t \geq 0\}$ the semimartingale

$$X^{\mu}_{\beta}(t) = x + \mu t + B(t) + (2\beta - 1)L^{a}_{t}(X^{\mu}_{\beta}). \tag{17}$$

To conclude this section, we study some invariance properties of asymmetric skew Bessel processes. To start with, $BES^{(\nu)}_{\beta}$ has the Brownian scaling property. Indeed, for every $c > 0$, the process $R'^{\nu}_{\beta}(t) = \sqrt{c}R^{\nu}_{\beta}(t/c)$ is a $BES^{(\nu)}_{\beta}$ with $a' = a\sqrt{c}$. Note that the invariance under time inversion no longer holds and we can prove that the process $tR^{\nu}_{\beta}(1/t)$ is a $BES^{(\nu)}_{\beta}$ only if $\beta = 1/2$. With similar arguments as in the proof of Lemma 4, we can also verify that for $a > 0$

$$\left(R^{\nu}_{\beta}(t)\right)^{1/q} = R^{q\nu}_{\beta}\left(q^2 \int_0^t \left(R^{\nu}_{\beta}(s)\right)^{-2/p} ds\right)$$

where R^{ν}_{β} and $R^{q\nu}_{\beta}$ are two asymmetric skew Bessel processes of order ν and $q\nu$ such that $p^{-1} + q^{-1} = 1$. We refer to Proposition (1.11) in Revuz and Yor (1998, p447) for a proof in case $\beta = 1/2$.

3 Radial property

To justify the term *Bessel*, we need to represent the asymmetric skew Bessel processes of dimension $d = 2, 3, \ldots$ as the distance to the origin of some multi-dimensional skew BM. We find it more convenient to investigate this property in the setting of generalized diffusions as defined by Portenko (1976) and (1979). Note that the restriction that the radial part of a multi-dimensional (generalized) diffusion remains a (generalized) diffusion is very restrictive. We start to recall briefly the construction of those processes.

A homogenous Markov process $X = \{X(t), t \geq 0\}$ is a diffusion if the limits

$$\begin{aligned} \lim_{t\to 0} \frac{1}{t} E_x\left[(X_t - x, \theta)^2\right] &= (a(x)\theta, \theta) \\ \lim_{t\to 0} \frac{1}{t} E_x[(X_t - x, \theta)] &= (b(x), \theta) \end{aligned} \tag{18}$$

exist for all x in the state space and $\theta \in \mathbb{R}^d$ where $(.,.)$ is the Euclidean scalar product. The limit $a(x)$ is called the diffusion matrix and $b(x)$ is called the drift. However, it is possible to construct diffusion where the limits (18) may be locally unbounded or even generalized functions. Portenko splits the d-dimensional Euclidean space $\mathbb{R}^d$ in two parts separated by a closed surface S: the interior region D and the exterior region $\mathbb{R}^d \setminus D$. He assumes also that at each point x of S there exists a tangent plan. $\nu(x)$ is the outer normal to S at the point $x \in S$. The theory of Portenko makes it possible to consider diffusion processes whose drift is a sum of a *regular* term $b(x)$ (that may be locally unbounded) and a term of the form $N(x)\delta_S(x-a)$ where $N(x)$ is a vector field and $\delta_S(x)$ is a generalized function such that

$$\int_{\mathbb{R}^d} \phi(x)\delta_S(x)dx = \int_S \phi(x)d\sigma$$

for every function $\phi(x) \in C_0(\mathbb{R}^d)$ with compact support and the *r.h.s.* integral is a surface integral. Portenko gives sense to the limits (18) in the dual of $C_0(\mathbb{R}^d)$ and defines the generalized diffusion as follows

Definition 3 *A homogeneous Markov process with transition density w.r.t the Lebesgue measure given by $p(t; x, y)$ and state space I is called a generalized diffusion process if the following conditions are satisfied for any $\epsilon > 0$:*

i) $lim_{t\to 0} \int_{\mathbb{R}^d} \phi(x) \left(\frac{1}{t}\int_{|x-y|\geq\epsilon} p(t; x, y)dy\right) dx = 0$

ii) $lim_{t\to 0} \frac{1}{t}\int_{\mathbb{R}^d} \phi(x) E_x\left[(X_t - x, \theta)\right] dx = \int_{\mathbb{R}^d} \phi(x)(b(x), \theta)dx$
$+ \int_S \phi(x)(N(x), \theta)q(x)d\sigma$

iii) $lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}^d}\phi(x)E_x\left[(X_t-x,\theta)^2\right]dx=\int_{\mathbb{R}^d}\phi(x)(a(x)\theta,\theta)dx$

for every function $\phi(x)\in C_0(\mathbb{R}^d)$ *with compact support included in the interior of* I, *where* $N(x)=a(x)\nu(x)$ *is the conormal and* $q(x)$ *is a continuous function on* $\mathbb{R}^d$ *with* $|q(x)|\le 1$ *for all* $x\in S$.

Definition 3 is uneasy to handle. Fortunately, Portenko proves that the associated semi-group of operators $\{T_t,t\ge 0\}$

$$T_t\phi(x)=\int_{\mathbb{R}^d}\phi(y)p(t;x,y)dy \tag{19}$$

acting on the set of bounded measurable functions $\phi(y)$, solves an integro-differential equation. We first define the normal derivative in $x\in S$ as

$$\nabla_x v(x)=\sum_{i,j=1}^{d}a_{ij}(x)\nu_j(x)\frac{\partial v}{\partial x_i}(x).$$

An interesting property of the semi-group $\{T_t,t\ge 0\}$ induced by the term in the drift proportional to the generalized function, is the discontinuity on the surface S of the normal derivative of $T_t\phi(x)$. The normal derivative of $T_t\phi(x)$ for $t>0$ is discontinuous for all $x\in S$ and for every bounded measurable function $\phi(x)$, Portenko (1979) proves that

$$(1-q(x))\nabla_x T_t\phi(x-)=(1+q(x))\nabla_x T_t\phi(x+).$$

Under some regularity properties upon the diffusion matrix $a(x)$, Portenko constructs the semi-group $\{T_t,t\ge 0\}$ by the formula

$$\begin{aligned}T_t\phi(x) &= T_t^0\phi(x) \qquad (20)\\ &+\int_0^t ds\int_S g(t-s,x,z)\frac{1}{2}\left(\nabla_x T_s\phi(z-)+\nabla_x T_s\phi(z+)\right)q(z)d\sigma_z\end{aligned}$$

where $T_t^0\phi(x)=\int_{\mathbb{R}^d}g(t;x,y)\phi(y)dy$ is the semi-group of the diffusion with drift $b(x)$ and diffusion matrix $a(x)$, and verifies that the conditions of definition 3 are satisfied.

The previous relation establishes clearly the connection between the BES^d_β constructed from its discontinuous speed and scale densities and the generalized diffusions treated by Portenko. Indeed, the transition density *w.r.t.* the Lebesgue measure of the BES^d_β satisfies the backward Fokker-Planck equation

$$\frac{\partial p}{\partial t}(t;x,.)=\frac{1}{2}\frac{d^2p}{dx^2}(t;x,.)+\frac{d-1}{2x}\frac{dp}{dx}(t;x,.)$$

subject to the condition $(1-\beta)\frac{dp}{dx}(t;a-,y) = \beta\frac{dp}{dx}(t;a+,y)$. It is then elementary to check that $T_t\phi(x) = \int p(t;x,y)\phi(y)dy$ satisfies equation (20) with $b(x) = (d-1)/2x$, $a(x) = 1$, $S = \{a\}$ and $q(x) = 2\beta - 1$. We are now able to recover the radial property of $d-$dimensional Bessel processes.

Theorem 1 *Let W_β be the generalized diffusion of dimension d defined by*

1. $a(x) := I_{d\times d}$, $b(x) := 0$
2. *S is the sphere of radius a in $\mathbb{R}^d$ centered to the origin,*
3. $q(x) := (2\beta - 1)$ *and* $|q(x)| \le 1$,
4. $N(x) := \left(\frac{x_1}{\|x\|}, \ldots, \frac{x_d}{\|x\|}\right) = \nu(x)$.

The distance to the origin

$$R_\beta^{(d)}(t) = \|W_\beta(t)\| = \sqrt{\left(W_\beta(t)^{(1)}\right)^2 + \ldots + \left(W_\beta(t)^{(d)}\right)^2}$$

is a BES_β^d.

Proof. A heuristic proof of the Lemma is to apply *Itô formula* to the function $f(x) := \|x\|$ for which

$$\begin{aligned} \frac{\partial f}{\partial x_i}(x) &= \frac{x_i}{\|x\|} \\ \frac{\partial^2 f}{\partial x_i \partial x_j}(x) &= \frac{\delta_{ij}}{\|x\|} - \frac{x_i x_j}{\|x\|^d} \end{aligned}$$

and bravely consider the drift of W_β as the generalized function $q(x)N(x)$ $\delta(\|x\| - a)$ and the drift of $R_\beta^{(d)}$ as $(d-1)/2x + (2\beta-1)\delta(x-a)$ where $\delta(.)$ is a delta Dirac function. As the Itô lemma was not investigated by Portenko, a rigorous proof starts with equation (20).

The semi-group $\{T_t^{(d)}, t \ge 0\}$ of the generalized diffusion W_β acting on the set of bounded $\mathcal{B}(\mathbb{R}^d)$-measurable functions $\phi(y)$ is solution of

$$\begin{aligned} T_t^{(d)}\phi(x) &= T_t^{(d),0}\phi(x) + (2\beta - 1) \\ &\quad \times \int_0^t ds \int_S g^{(d)}(s,x,z)\frac{1}{2}\left(\nabla_x T_{t-s}^{(d)}\phi(z-) + \nabla_x T_{t-s}^{(d)}\phi(z+)\right) d\sigma_z \end{aligned}$$

where $T_t^{(d),0}\phi(x) = \int_{\mathbb{R}^d} g^{(d)}(t;x,y)\phi(y)dy$ is the semi-group of the d-dimensional Brownian motion. We construct the semi-group $\{T_t, t \ge 0\}$ of the BES_β^d

acting on bounded $\mathcal{B}([0,+\infty))$-measurable functions $\varphi(r)$ taking the semigroup of the BES^d for $T_t^0\varphi(r)$. The function $\phi(x)=\varphi(\|x\|)$ is a bounded $\mathcal{B}(\mathbb{R}^d)$-measurable function and the following relation holds

$$\begin{aligned}\nabla_r T_t\varphi(r) &= \sum_{i=1}^{d}\frac{x_i}{\|x\|}\frac{\partial T_t^{(d)}\varphi}{\partial x_i}(\|x\|),\\ &= \sum_{i=1}^{d}\nu_i(x)\frac{\partial T_t^{(d)}\phi}{\partial x_i}(x),\\ &= \nabla_x T_t^{(d)}\phi(x).\end{aligned}$$

The result follows from the previous equality as the BES^d is the radial part of the $d-$dimensional BM and $S=\{x\in\mathbb{R}^d | r=\|x\|=a\}$. □

4 Factorizations

The theory of Bessel processes has led in the long run to various applications. It provides a probabilistic alternative for problems that often reduce to solving a tedious partial differential equation. Among many others, we can cite the distribution of the Wiener functional $A^\nu(t)=\int_0^t e^{2(B(s)+\nu s)}ds$. We refer to De Schepper et al. (1992) for the Feynman-Kac approach to this distribution and to Yor *e.g.* (1992), (1993) and (2001) for the connection with Bessel processes. In this section, we propose two useful factorizations that involve time changed (asymmetric skew) Bessel processes.

The Lamperti factorization states that a geometric BM is a time changed Bessel process

$$e^{B(t)+\nu t}=R^\nu\left(A^\nu(t)\right) \tag{21}$$

where R^ν is a Bessel process and $A^\nu(t)=\int_0^t e^{2(B(s)+\nu s)}ds$. It establishes clearly the connection between Bessel processes and the distribution of $A^\nu(t)$. This relation has received much attention, we can cite Yor (2001) for more general diffusions and Lamperti (1972) for exponential of a Lévy process. The next lemma states that similar relation holds between skew BM with drift and asymmetric skew Bessel processes.

Lemma 5 *Let X_β^ν be a BM skewed at a with drift ν and skew parameter β, then*

$$e^{X_\beta^\nu(t)}=R_\beta^\nu\left(A^\nu(t)\right)$$

where $A^\nu(t) = \int_0^t e^{2X^\nu_\beta(s)}ds$ *and* R^ν_β *is a* $BES^{(\nu)}_\beta$ *skewed at* e^a *and starting at* e^x*.*

Proof. We first consider the case $\nu \geq 0$. We use similar arguments as in Carmona and Petit (1994) paying attention to the restriction on the domains. It is now clear that the skew BM with drift ν is a linear diffusion with generator $\mathcal{G}^{(1)} = \frac{1}{2}\frac{d^2}{dx^2} + \nu\frac{d}{dx}$ acting on the domain $\mathcal{D}^{(1)} = \{f : f, \mathcal{G}^{(1)}f \in C_b(\mathbb{R}), (1-\beta)f'(a-) = \beta f'(a+)\}$. Similarly, the $BES^{(\nu)}_\beta$ is also a linear diffusion with generator $\mathcal{G}^{(2)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2\nu+1}{2x}\frac{d}{dx}$ acting on $\mathcal{D}^{(2)} = \{f : f, \mathcal{G}^{(2)}f \in C_b([0,+\infty)), \frac{df^+}{ds}(0+) = 0, (1-\beta)f'(e^a-) = \beta f'(e^a+)\}$. Thus, for any function f such that $f \circ e^x$ is in the domain of the generator of X^ν_β, we have

$$\begin{aligned} f(R^\nu_\beta(t)) - f(R^\nu_\beta(0)) &= (f \circ e^x)(X^\nu_\beta(\tau(t))) - (f \circ e^x)(X^\nu_\beta(0)) \\ &= M(\tau(t)) + \int_0^{\tau(t)} \mathcal{G}^{(1)}(f \circ e^x)(X^\nu_\beta(s))ds \end{aligned}$$

where $\tau(t) = \inf\{s | A^\nu(s) > t\}$ and $M(\tau(t))$ is a local martingale. Implementing the time change as proposed by Carmona and Petit (1994), we obtain

$$\begin{aligned} f(R^\nu_\beta(t)) - f(R^\nu_\beta(0)) &= M(\tau(t)) + \int_0^t \mathcal{G}^{(1)}(f \circ e^x)(X_\beta(\tau(s)))\frac{ds}{e^{2X^\nu_\beta(\tau(s))}} \\ &= M(\tau(t)) + \int_0^t \mathcal{G}^{(1)}(f \circ e^x)(\ln R^\nu_\beta(s))\frac{ds}{(R^\nu_\beta(s))^2} \end{aligned}$$

By identification, we can conclude that the generator of R^ν_β is $\mathcal{G}^{(2)}$. As the only restriction on f is that $f \circ e^x$ belongs to $\mathcal{D}^{(1)}$, we can verify that $\mathcal{G}^{(2)}f \in C_b([0,\infty))$ and $(1-\beta)f'(e^a-) = \beta f'(e^a+)$. As $-\infty$ is a natural boundary for the skew BM with drift $\nu \geq 0$, we obtain the condition $\frac{df^+}{ds}(0+) = 0$. When $\nu < 0$, the same factorization holds for $t < +\infty$ and do not involve the nature of the left boundary 0. $\square$

The following factorization will be useful in the study of the exponential functional

$$\hat{A}^\nu(t) = \int_0^t e^{2B^\nu(s)}\sigma^2_\beta(B^\nu(s))ds \tag{22}$$

where $B^\nu(s) = x + \nu s + B(s)$ and $\sigma_\beta(x) = (1-\beta)1_{x\geq 0} + \beta 1_{x<0}$. It can be interpreted as a generalization of the Brownian time scaling property of Bessel processes.

Lemma 6 *Let B^ν be a BM with drift ν, then*

$$e^{B^\nu(t)} = \hat{R}^\nu_\beta\left(\hat{A}^\nu(t)\right)$$

where $\hat{A}^\nu(t) = \int_0^t e^{2B^\nu(s)}\sigma^2_\beta(B^\nu(s))ds$ and $\hat{R}^\nu$ is a time changed $BES^{(\nu)}$

$$\hat{R}^\nu_\beta(t) = R^\nu\left(\int_0^t ds/\sigma^2_\beta(\ln R^\nu(s))\right) \tag{23}$$

starting from e^x at time 0.

Proof. We start with the case $\nu \geq 0$. Let $r_\beta(x)$ be the following continuous function

$$r_\beta(x) = \begin{cases} (1-\beta)x + \beta, & x \geq 1 \\ \beta x + (1-\beta), & x < 1, \end{cases}$$

an application to the process $Y(t) = r_\beta\left(e^{B^\nu(t)}\right)$ of Tanaka fomula yields

$$dY(t) = e^{B^\nu(t)}\sigma_\beta\left(B^\nu(t)\right)\left(\frac{2\nu+1}{2}dt + dB(t)\right) + \frac{1-2\beta}{2}dL^0_t(B^\nu).$$

We change the clock of the process $Y(t)$ using the inverse function $\tau(t) = \inf\{s|\hat{A}^\nu(s) > t\}$ to obtain

$$Y(\tau(t)) - Y(0) = \frac{2\nu+1}{2}\int_0^t b(Y(\tau(s)))ds + B(t) + \frac{(1-2\beta)}{2}L^0_{\tau(t)}(B^\nu)$$

where the function $b(x)$ is defined in (9). Using the properties $L^0_{\tau(t)}(B^\nu) = L^0_t(B^\nu(\tau))$ and $L^{f(0)}_t(Y) = f'(0)L^0_t(B^\nu)$ with $f(x) = r_\beta(e^x)$, we conclude that

$$dY(\tau(t)) = \frac{2\nu+1}{2}b(Y(\tau(t)))dt + dB(t) + (1-2\beta)dL^1_t(Y(\tau)).$$

By standard use of Tanaka formula, we can show that $Y(\tau)$ is equal in law to $r_\beta(\hat{R}^\nu)$ and thus

$$r_\beta\left(e^{B^\nu(t)}\right) = r_\beta\left(\hat{R}^\nu_\beta\left(\hat{A}^\nu(t)\right)\right).$$

The bijectivity of the function $r_\beta(.)$ completes the proof. For $\nu < 0$, one can prove the same relation by comparing the generators as done in the proof of Lemma 5. $\square$

5 Applications

Following Yor (1992), we can use the factorizations obtained in the previous section to relate the distribution of the perpetuities $A^\nu(+\infty)$ and $\hat{A}^\nu(+\infty)$ to the first hitting time of a diffusion. Let $H_0(Z) = \inf\{t > 0 | Z(t) = 0\}$ denote the first hitting time of the diffusion Z at the origin. To avoid straightforward complications, we assume that $\beta \in (0, 1)$.

Proposition 1 *The following identities in law hold for* $\nu < 0$

$$\begin{aligned} A^\nu(+\infty) &= H_0(R^\nu_\beta) \\ \hat{A}^\nu(+\infty) &= H_0(\hat{R}^\nu_\beta) \end{aligned}$$

where R^ν_β *is a* BES^ν_β *starting from* e^x *and* $\hat{R}^\nu_\beta$ *is a time changed* BES^ν *starting from* e^x*.*

Proof. We use similar arguments as Yor (1992). From the Lamperti representation $e^{X^\nu_\beta(t)} = R^\nu_\beta\left(A^\nu(t)\right)$, we deduce that

$$\begin{aligned} \lim_{t \to +\infty} e^{X^\nu_\beta(t)} &= R^\nu_\beta\left(A^\nu(+\infty)\right) \\ &= 0, \end{aligned}$$

as $-\infty$ is an attracting boundary for the skew BM with drift $\nu < 0$ and $\beta \in [0, 1)$. Moreover e^x is a strictly monotone and increasing function, we conclude that $A^\nu(+\infty) = H_0(R^\nu_\beta)$. The second part of the result follows from Lemma 6. □

We can obtain the Laplace transform of those perpetuities by means of the fundamental solutions associated to R^ν_β and $\hat{R}^\nu_\beta$. Indeed, see Itô and McKean (1974),

$$E_x\left[e^{-\alpha H_y(Z)}\right] = \begin{cases} \frac{\psi_\alpha(x)}{\psi_\alpha(y)}, & x \le y \\ \frac{\varphi_\alpha(x)}{\varphi_\alpha(y)}, & x \ge y \end{cases}$$

where ψ_α and φ_α are the fundamental solutions associated to the infinitesimal generator of Z. Note that the Laplace transform of $\hat{A}^\nu(+\infty)$ (to some normalizing factor) has already been obtained by Salminen and Yor (2004) using excursion theory.

A second application can be found in the pricing of *weighted* Asian option. We define the payoff of this contract as

$$\max\{A(t) - K, 0\} = (A(t) - K)^+$$

where $A(t)$ is the average of the underlying asset weighted depending whether the underlying asset is below some level a:

$$A(t) = \frac{w_1}{t}\int_0^t S(u)1_{S(u)<a}du + \frac{w_2}{t}\int_0^t S(u)1_{S(u)\geq a}du$$

for some $w_1, w_2 \geq 0$. The price $C_t(K)$ of this financial contract is the expectation under the risk neutral measure Q of the discounted payoff, $C_t(K) = e^{-rt}E^Q[(A(t) - K)^+]$ where r is the constant interest rate. When the underlying asset follows a geometric Brownian motion under the risk neutral measure, the problem reduces to finding the distribution of the functional

$$\hat{A}^\nu(t) = \int_0^t e^{2B^\nu(s)}\sigma_\beta^2(B^\nu(s))ds$$

for some ν and $\sigma_\beta(x) = (1-\beta)1_{x\geq 0} + \beta 1_{x<0}$. The distribution of the functional $\hat{A}^\nu(t)$ is difficult. Yor (1993) obtains the double Laplace transform using excursion theory as a pathwise alternative to the Feynman-Kac approach. In this paper, we provide additional insight into this quantity adapting the methods developed by Yor and other authors (2001). In what follows, the process $Z^\nu_\beta = r_\beta(\hat{R}^\nu_\beta)$ with $\hat{R}^\nu_\beta(t) = R^\nu\left(\int_0^t ds/\sigma^2_\beta(\ln R^\nu(s))\right)$ plays an important role. The process $Z^\nu_\beta = \{Z^\nu_\beta(t), t \geq 0\}$ is a solution of the SDE

$$dZ^\nu_\beta(t) = \frac{2\nu+1}{2}b(Z^\nu_\beta(t))dt + dB(t) + (1-2\beta)dL^1_t(Z^\nu_\beta), \tag{24}$$

where the function $b(x)$ is defined by equation (9). We denote by P^ν_a the law of Z^ν_β starting at $a \geq 1-\beta$ and by E^ν_a the expectation relative to P^ν_a. The following continuity relation will be useful.

Lemma 7 *Let $\mathcal{F}_t = \sigma(Z(s), s \leq t)$ be the filtration generated by the coordinate process Z, the Girsanov's Theorem implies the following continuity relation:*

$$P^\nu_a|_{(\mathcal{F}_t\cap(t<H_0(Z)))} = \left(\frac{r_\beta^{-1}(Z(t))}{r_\beta^{-1}(a)}\right)^\nu e^{-\frac{\nu^2}{2}\int_0^t b^2(Z(s))ds} \times P^0_a|_{\mathcal{F}_t}.$$

where $r_\beta(x) = \int^x \sigma_\beta(\ln y)dy$ and $r_\beta(1)$=1.

Proof. The Girsanov, Cameron-Martin's Theorem provides the Radon-Nikodym derivative of P^ν_a relative to P^0_a

$$\frac{dP^\nu_a}{dP^0_a} = e^{-\frac{\nu^2}{2}\int_0^t b^2(Z^0_\beta(s))ds + \nu\int_0^t b(Z^0_\beta(s))dB(s)}$$

An application of Tanaka formula yields

$$\begin{aligned}\frac{dP_a^\nu}{dP_a^0} &= e^{-\frac{\nu(\nu+1)}{2}\int_0^t b^2(Z_\beta^0(s))ds+\nu\int_0^t b(Z_\beta^0(s))dZ_\beta^0(s)+\frac{(2\beta-1)}{2}(b(1+)+b(1-))L_t^1(Z_\beta^0)} \\ &= e^{\nu\left(\bar{b}(Z_\beta^0(t))-\bar{b}(a)\right)}e^{-\frac{\nu^2}{2}\int_0^t b^2(Z_\beta^0(s))ds}\end{aligned}$$

with $\bar{b}(x)=\int^x b(y)dy$ since $(1-\beta)b(1+)=\beta b(1-)$. Straightforward computations complete the proof. $\square$

In the following Lemma, we relate the law of the exponential functional $\hat{A}^\nu$ taken at an independent exponential time T_λ to the time changed Bessel processes $\hat{R}_\beta^\nu$.

Proposition 2 *Let T_λ be an exponential random variable with parameter $\lambda>0$ independent of $\hat{A}^\nu$, then*

$$\begin{aligned}P\left(\hat{A}^\nu(T_\lambda)\geq u\right) &= \lambda\int_0^{+\infty}e^{-\lambda t}P\left(\hat{A}^\nu(t)\geq u\right)dt \\ &= E_1\left[\left(\hat{R}_\beta^\mu(u)\right)^{\nu-\mu}\right], \qquad (25)\end{aligned}$$

where $\mu=\sqrt{2\lambda+\nu^2}$.

Proof. To start with, we reverse the factorization $e^{B^\nu(t)}=\hat{R}_\beta^\nu(\hat{A}^\nu(t))$. Define $Y(t)=r_\beta(e^{B^\nu(t)})$, then

$$\begin{aligned}Y(t) &= Z_\beta^\nu\left(\hat{A}^\nu(t)\right) \\ &= Z_\beta^\nu\left(\int_0^t b^{-2}(Y(s))ds\right),\end{aligned}$$

or equivalently $Y\left(\int_0^t b^2(Z_\beta^\nu(s))ds\right)=Z_\beta^\nu(t)$. Using this time change relation, similarly to Yor (2001, p98), we obtain

$$\begin{aligned}P\left(\hat{A}^\nu(T_\lambda)\geq u\right) &= \lambda\int_0^{+\infty}e^{-\lambda t}P\left(\hat{A}^\nu(t)\geq u\right)dt \\ &= \lambda\int_0^{+\infty}e^{-\lambda t}P\left(H^\nu(u)\leq t\right)dt \\ &= E_1\left[e^{-\lambda H^\nu(u)}\right],\end{aligned}$$

where $H^\nu(u) = \int_0^u b^2(Z^\nu_\beta(s))ds$. The absolute continuity relation of Lemma 7 completes the proof:

$$\begin{aligned} E_1\left[e^{-\lambda H^\nu(u)}\right] &= E_1^0\left[\left(r_\beta^{-1}(Z(t))\right)^\nu e^{-\frac{\mu}{2}^2 H(u)}\right] \\ &= E_1\left[\left(\hat{R}^\mu_\beta(u)\right)^{\nu-\mu}\right]. \end{aligned}$$

□

We can obtain the Green function of the time changed Bessel processes $\hat{R}^\nu_\beta$ as a linear combination of modified Bessel functions. Unfortunately, no simple expressions exist for the transition density. When $\beta \neq 1$ or 0, the spectrum of the associated infinitesimal generator is continuous and the inversion of the Laplace transform is a priori tedious. Nevertheless, we find interesting to show that the methods developed by Yor (2001) still apply. The pevious results permit to interpret the weights w_1 and w_2 in terms of speed-up (and slow-down) of the time in the underlying Bessel processes. It is elementary to check that we can no longer rely on the additivity of the square of $\hat{R}^\mu_\beta$ to obtain a simple expression of the expectation. Nevertheless, following Yor (2001, p99), we can use the formula

$$\frac{1}{x^b} = \frac{1}{\Gamma(b)}\int_0^{+\infty} e^{-tx}t^{b-1}dt$$

where Γ is the Gamma function, to reformulate equation (25) as

$$E_1\left[\left(\hat{R}^\mu_\beta(u)\right)^{\nu-\mu}\right] = \frac{1}{\Gamma(\nu-\mu)}\int_0^{+\infty} E_1\left[e^{-t\hat{R}^\mu_\beta(u)}\right]t^{\nu-\mu-1}dt.$$

Further research will develop methods to compute series expansions for this expectation in the simplest cases $\beta = 1$ or $\beta = 0$. For instance, we can take profit of the following Tanaka formula

$$\int_0^t 1_{R^\mu(s)>1}dR^\mu(s) = (R^\mu(t)-1)^+ - \frac{1}{2}L^1_t(R^\mu).$$

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Corresponding author:

Marc Decamps
Katholieke Universiteit Leuven, Department of Applied Economics
Naamsestraat 69, 3000 Leuven, Belgium
Tel: +32 16 326770
e-mail: marc.decamps@econ.kuleuven.ac.be

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