Premium Calculation and Insurance Pricing

Roger J.A. Laeven* — Tilburg University and CentER

Marc J. Goovaerts — Catholic University of Leuven and University of Amsterdam

First edition (published): May, 2007

Third edition: July, 2011

Abstract

This survey of premium calculation and insurance pricing explains classical theories and their recent generalizations, summarizes main issues and results, and describes current developments in the area.

Keywords: Premium Calculation Principles; Expected Utility; Equivalent Utility; Equilibrium Pricing; Pareto Optimal Risk Exchange; Arbitrage-free Pricing; Pricing Kernel; Deflator; Esscher Premium; Exponential Premium; Rank-dependent Expected Utility

JEL-Classification: D81, G10, G20

MSC-Classification: 62P05

*Corresponding author. E-mail: `R.J.A.Laeven@uvt.nl`, Phone: +31 13 466 2430, Fax: +31 13 466 3280.

1 Introduction

The price of insurance is the monetary value for which two parties agree to exchange risk and "certainty". There are two commonly encountered situations in which the price of insurance is subject of consideration: when an individual agent (for example, a household), bearing an insurable risk, buys insurance from an insurer at an agreed periodic premium; and when insurance portfolios (that is, a collection of insurance contracts) are traded in the financial industry (e.g., being transferred from an insurer to another insurer or from an insurer to the financial market (: securitization)).

Pricing in the former situation is usually referred to as *premium calculation* while pricing in the latter situation is usually referred to as *insurance pricing*, although such a distinction is not unambiguous. Below we refrain from such an explicit distinction although the reader may verify that some of the methods discussed are more applicable to the former situation and other methods are more applicable to the latter situation.

To fix our framework, we state the following definitions and remarks:

Definition 1 (Fundamentals) *We fix a measurable space* $(\Omega, \mathcal{F})$ *where* Ω *is the outcome space and* $\mathcal{F}$ *is a (*σ*-)algebra defined on it. A risk is a random variable defined on* $(\Omega, \mathcal{F})$*; that is,* $X : \Omega \to \mathbb{R}$ *is a risk if* $X^{-1}((-\infty, x]) \in \mathcal{F}$ *for all* $x \in \mathbb{R}$*. A risk represents the final net loss of a position (contingency) currently held. When* $X > 0$*, we call it a loss, whereas when* $X \leq 0$*, we call it a gain. The class of all random variables on* $(\Omega, \mathcal{F})$ *is denoted by* $\mathcal{X}$*.*

Definition 2 (Premium calculation principle (pricing principle)) *A premium (calculation) principle or pricing principle* π *is a functional assigning a real number to any random variable defined on* $(\Omega, \mathcal{F})$*; that is,* π *is a mapping from* $\mathcal{X}$ *to* $\mathbb{R}$*.*

Remark 3 *In general, no integrability conditions need to be imposed on the elements of* $\mathcal{X}$*. In the absence of integrability conditions, some of the premium principles studied below will be infinite for subclasses of* $\mathcal{X}$*. Instead of imposing integrability conditions, we may extend the range in the definition of* π *to* $\mathbb{R} \cup \{-\infty, +\infty\}$*. In case* $\pi[X] = +\infty$*, we say that the risk is unacceptable or non-insurable.*

Remark 4 *Statements and definitions provided below hold for all* $X, Y \in \mathcal{X}$ *unless mentioned otherwise.*

No attempt is made to survey all of the many aspects of premium calculation and insurance pricing. For example, we will largely ignore contract theoretical issues, we will largely neglect insurer expenses and profitability as well as solvency margins.

In the recent literature, premium principles are often studied under the guise of risk measures; see Föllmer & Schied (2004) and Denuit *et al.* (2006) for recent reviews of the risk measures literature.

2 Classical Theories and Some Generalizations

Classical actuarial pricing of insurance risks mainly relies on the economic theories of decision under uncertainty, in particular on Von Neumann & Morgenstern's (1947) expected utility theory and Savage's (1954) subjective expected utility theory; the interested reader is referred to the early monographs of Borch (1968, 1974), Bühlmann (1970), Gerber (1979) and Goovaerts, De Vylder & Haezendonck (1984) for expositions and applications of this theory in insurance economics and actuarial science. Using the principle of equivalent utility and specifying a utility function, various well-known premium principles can be derived. An important example is the exponential premium principle, obtained by using a (negative) exponential utility function, having a constant rate of risk aversion.

2.1 The Principle of Equivalent Utility

Consider an agent who is carrying a certain risk. Henceforth he will be referred to as the "insured" even though he might decide not to purchase the insurance policy he is offered. Furthermore, consider an insurance company offering an insurance policy for the risk.

We consider below a simple one-period-one-contract model, in which the insurer's wealth at the beginning of the period is w and the loss incurred due to the insured risk during the period is X, so that the insurer's wealth at the end of the period is $w+\pi[X]-X$. The loss X is uncertain and is here represented by a non-negative random variable with some distribution $\mathbb{P}[X \leq x]$. We assume that both the insurer and the insured have preferences that comply with the expected utility hypothesis.

Let us first consider the viewpoint of the insurer. For a given utility function $u : \mathbb{R} \rightarrow \mathbb{R}$ we consider the expected utility $\mathbb{E}[u(w + \pi[X] - X)]$. In the Von Neumann-Morgenstern framework, the utility function u is subjective, whereas the probability measure is assumed to be objective, known and given. In the more general framework of Savage (1954), also the probability measure can be subjective. In the latter framework, the probability measure need not be based on objective statistical information, but may be based on subjective judgements of the decision situation under consideration, just as u is.

From the expected utility expression, an *equivalent utility principle* (also known under the slight misnomer *zero utility principle*) can be established as follows: for a given random variable X and a given real number w, the equivalent utility premium $\pi^-[X]$ of the insurer is derived by solving

$$\mathbb{E}[u(w + \pi[X] - X)] = u(w). \tag{1}$$

Under the assumption that u is continuous (which is implied by the usual set of axioms characterizing expected utility preferences) and provided that the expectation is finite, a solution exists. If the insurer is risk-averse (which for expected utility preferences is equivalent to the utility function being concave), one easily proves that $\pi^-[X] \geq \mathbb{E}[X]$. The insurer will sell the insurance if and only if he can charge a premium $\pi[X]$ that satisfies $\pi[X] \geq \pi^-[X]$.

Next, we consider the viewpoint of the insured. An insurance treaty that for a given random variable X leaves the insured with final wealth $\bar{w} - \pi[X]$, will be preferred to full self insurance, which leaves the insured with final wealth $\bar{w}-X$, if and only if $\bar{u}(\bar{w}-\pi[X]) \geq \mathbb{E}[\bar{u}(\bar{w}-X)]$, where $\bar{u}$ denotes the utility function of the insured. The equivalent utility premium $\pi^+[X]$ is derived by solving

$$\bar{u}(\bar{w}-\pi[X]) = \mathbb{E}[\bar{u}(\bar{w}-X)].$$

The insured will buy the insurance if and only if $\pi[X] \leq \pi^+[X]$. One easily verifies that a risk-averse insured is willing to pay more than the pure net premium $\mathbb{E}[X]$. An insurance treaty can be signed both by the insurer and by the insured only if the premium satisfies $\pi^-[X] \leq \pi[X] \leq \pi^+[X]$.

Example 5 *Consider an agent whose preferences can be described by an exponential utility function*

$$u(x) = \frac{1}{\alpha}\left(1 - e^{-\alpha x}\right), \qquad \alpha > 0, \tag{2}$$

where α is the coefficient of absolute risk aversion. We find that

$$\pi^+[X] = \frac{1}{\alpha}\log(\mathrm{m}_X(\alpha)), \tag{3}$$

where $\mathrm{m}_X(\alpha)$ is the moment generating function of X evaluated in α. We note that for this specific choice of the utility function, $\pi^+[X]$ does not depend on w. Furthermore, notice that the expression for $\pi^-[X]$ is similar. However, now α corresponds to the risk aversion of the insurer. When $\alpha \downarrow 0$, (i.e., for a risk-neutral agent) the exponential premium reduces to $\mathbb{E}[X]$.

Different specifications of the utility function yield different expressions for $\pi^+[X]$ and $\pi^-[X]$. It is not guaranteed that an explicit expression for $\pi^+[X]$ or $\pi^-[X]$ is obtained. In a competing insurance market, the insurer may want to charge the lowest "feasible" premium, in which case π will be set equal to π^-.

2.1.1 Rank-dependent expected utility and Choquet expected utility

Since the axiomatization of expected utility theory by Von Neumann & Morgenstern (1947) and Savage (1954) numerous objectives have been raised against it; see e.g., Allais (1953). Most of these relate to the descriptive power of the theory, that is, to empirical evidence of the extent to which agents' behavior under risk and uncertainty coincides with expected utility theory.

Motivated by empirical evidence that individuals often violate expected utility, various alternative theories have emerged, usually termed as non-expected utility theories; see Sugden (1997) for a review.

A prominent example is the *rank-dependent expected utility* (RDEU) theory introduced by Quiggin (1982) under the guise of *anticipated utility theory*, and by Yaari (1987) for the special case of linear utility. A more general theory for decision under uncertainty (rather than decision under risk), which is known as *Choquet expected utility theory* (CEU), is due to Schmeidler (1989); see also Choquet (1953-4), Greco (1982), Schmeidler (1986) and Denneberg (1994).

Consider an economic agent who is to decide whether or not to hold a lottery (prospect) denoted by the random variable V. Under a similar set of axioms as used to axiomatize expected utility theory but with a modified additivity axiom, Yaari (1987) showed that the amount of money c at which the decision maker is indifferent between holding the lottery V or receiving the amount c with certainty can be represented as (in his original work, Yaari (1987) restricts attention to random variables supported on the unit interval; also, Yaari (1987) imposes a strong continuity axiom to ensure that g is continuous; we present here a more general version):

$$c = -\int_{-\infty}^{0} \left(1 - g(\bar{F}_V(v))\right) dv + \int_{0}^{+\infty} g(\bar{F}_V(v))dv, \tag{4}$$

with $\bar{F}_V(v) := 1 - \mathbb{P}[V \leq v]$. Here, the function $g : [0,1] \to [0,1]$ is non-decreasing and satisfies $g(0) = 0$ and $g(1) = 1$. It is known as a *distortion function.* Provided that g is right-continuous, that $\lim_{x\to+\infty} xg(\bar{F}(x)) = 0$ and that $\lim_{x\to-\infty} x\left(1 - g(\bar{F}(x))\right) = 0$, expression (4) is an expectation calculated using the distorted distribution function $F_g^*(x) := 1 - g(\bar{F}(x))$, that is, $c = \int_{-\infty}^{+\infty} xdF_g^*(x)$. In the economics literature, c is usually referred to as the *certainty equivalent.* Applying the equivalent utility principle within the RDEU framework amounts to solving

$$\int_{-\infty}^{+\infty} (w + \pi[X] - X)dF_g^*(x) = w, \tag{5}$$

from which we obtain the *distortion premium principle*:

$$\pi^-[X] = \int_{-\infty}^{+\infty} xdF_{\bar{g}}^*(x), \tag{6}$$

with $\bar{g}(p) = 1 - g(1-p)$, $0 \leq p \leq 1$, being the *dual distortion function.* The premium principle (6) and its appealing properties were studied by Wang (1996) and Wang, Young & Panjer (1997).

Example 6 (PH-distortion) *Consider as an example the proportional hazard (PH) distortion function given by $\bar{g}(x) = x^{1/\alpha}$, $\alpha \geq 1$, advocated by Wang (1996) and Wang, Young & Panjer (1997). The value of the parameter α determines the degree of risk aversion: the larger the value of α, the larger the risk aversion with $\alpha = 1$ corresponding to the non-distorted (base) case. In this case,*

$$\pi^-[X] = -\int_{-\infty}^{0} \left(1 - (\bar{F}_X(x))^{1/\alpha}\right) dx + \int_{0}^{+\infty} (\bar{F}_X(x))^{1/\alpha}dx. \tag{7}$$

The decumulative distribution function (or tail distribution function) $\bar{F}_X$ provides a natural insight in how to distribute insurance premiums between different layers of insurance since the net premium for a layer $(a, a+b]$ can be computed as

$$\int_{a}^{a+b} \bar{F}_X(x)dx. \tag{8}$$

An important concept in the axiomatization of RDEU as well as CEU is *comonotonicity.* We state the following definition:

Definition 7 (Comonotonicity) *A pair of random variables $X, Y : \Omega \to (-\infty, +\infty)$ is comonotonic if*

(i) there is no pair $\omega_1, \omega_2 \in \Omega$ such that $X(\omega_1) < X(\omega_2)$ while $Y(\omega_1) > Y(\omega_2)$;

(ii) there exists a random variable $Z : \Omega \to (-\infty, +\infty)$ and non-decreasing functions f, g such that

$$X(\omega) = f(Z(\omega)), \qquad Y(\omega) = g(Z(\omega)), \qquad \text{for all } \omega \in \Omega. \tag{9}$$

As we will see below, (6) is additive for sums of comonotonic risks.

Another prominent example of a non-expected utility theory is *maximin expected utility theory* introduced by Gilboa & Schmeidler (1989). It involves the evaluation of worst case scenarios. Laeven (2005) proposes a theory that unifies CEU and maximin expected utility theory for linear utility.

2.2 Equilibrium Pricing and Pareto Optimality

Bühlmann (1980) argued that in contrast to classical premium principles which depend on characteristics of the risk on its own, it is more realistic to consider premium principles which take into account general market conditions (e.g., aggregate risk, aggregate wealth, dependencies between the individual risk and general market risk). Bühlmann derived an *economic premium principle* in which a functional $\pi[\cdot, \cdot]$ assigns to a random variable X representing the risk to be insured and a random variable Z representing general market risk, a real number π; that is, $\pi : (\mathcal{X}, \mathcal{Z}) \to \mathbb{R}$. We will briefly review Bühlmann's economic premium principle; it is based on general equilibrium theory, a main branch in microeconomics.

Consider a pure exchange market with n agents. Think of the agents as being buyers or sellers of insurance policies. Each agent is characterized by

1. a utility function $u_i(x)$, with $u_i'(x) > 0$ and $u_i''(x) \leq 0$, $i = 1, \ldots, n$;

2. an initial wealth w_i, $i = 1, \ldots, n$.

Each agent faces an exogenous potential loss according to an individual risk function $X_i(\omega)$, representing the payment due by agent i if state ω occurs. In terms of insurance, $X_i(\omega)$ represents the risk of agent i before (re)insurance. Furthermore, each agent buys a *risk exchange* denoted by an individual risk exchange function $Y_i(\omega)$, representing the payment received by agent i if ω occurs. The market price of Y_i is denoted by $\pi[Y_i, \cdot]$. Bühlmann used the concept of a pricing density (pricing kernel, state price deflator) $\varphi : \Omega \to \mathbb{R}$ so that

$$\pi[Y_i, \cdot] = \int_\Omega Y_i(\omega)\varphi(\omega)d\mathbb{P}(\omega), \tag{10}$$

for a fixed probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. An equilibrium is then defined as the pair $(\varphi^*, \mathbf{Y}^*)$ for which:

1.

$$\forall i, \quad \int_\Omega u_i \left(w_i - X_i(\omega) + Y_i^*(\omega) - \int_\Omega Y_i^*(\omega')\varphi^*(\omega')d\mathbb{P}(\omega') \right) d\mathbb{P}(\omega) : \tag{11}$$

maximal for all possible choices of the exchange function $Y_i(\omega)$;

2.

$$\sum_{i=1}^n Y_i^*(\omega) = 0, \qquad \forall \omega \in \Omega. \tag{12}$$

φ^* is called the equilibrium pricing density and $\mathbf{Y}^*$ is called the equilibrium risk exchange.

Assume that the utility function of agent i is given by

$$\frac{1}{\alpha_i}\left(1 - e^{-\alpha_i x}\right),$$

where $\alpha_i > 0$ denotes the absolute coefficient of risk aversion of agent i and $\frac{1}{\alpha_i}$ is his risk tolerance. Solving the equilibrium conditions, Bühlmann obtained the following expression for the equilibrium pricing density:

$$\varphi^*(\omega) = \frac{e^{\alpha Z(\omega)}}{\mathbb{E}[e^{\alpha Z}]}; \qquad Z(\omega) = \sum_{i=1}^n X_i(\omega). \tag{13}$$

Equation (13) determines an economic premium principle for any random loss X, namely

$$\pi[X, Z] = \frac{\mathbb{E}[Xe^{\alpha Z}]}{\mathbb{E}[e^{\alpha Z}]}. \tag{14}$$

Notice that if X and $Z - X$ are independent a *standard* premium principle is obtained:

$$\pi[X, Z] = \frac{\mathbb{E}[Xe^{\alpha X}]\mathbb{E}[e^{\alpha(Z-X)}]}{\mathbb{E}[e^{\alpha X}]\mathbb{E}[e^{\alpha(Z-X)}]} = \frac{\mathbb{E}[Xe^{\alpha X}]}{\mathbb{E}[e^{\alpha X}]} = \pi[X]. \tag{15}$$

This is the time-honored *Esscher premium*. Here, $\frac{1}{\alpha} = \sum_{i=1}^n \frac{1}{\alpha_i}$ can be interpreted as the risk tolerance of the market as a whole. Note that if the insurance market would contain a large number of agents, the value of $\frac{1}{\alpha} = \sum_{i=1}^n \frac{1}{\alpha_i}$ would be large, or equivalently, the value of α would be close to zero and hence applying the Esscher principle would yield the net premium $\mathbb{E}[X]$.

Remark 8 *In equilibrium only systematic risk requires a risk loading. To see this, suppose that a risk X can be decomposed as follows:*

$$X = X_s + X_n, \tag{16}$$

where X_s is the systematic risk which is comonotonic with Z, and X_n is the non-systematic or idiosyncratic risk which is independent of Z (and X_s). Substituting this decomposition in (14), we obtain

$$\pi[X, Z] = \mathbb{E}[X_n] + \frac{\mathbb{E}[X_s e^{\alpha Z}]}{\mathbb{E}[e^{\alpha Z}]}. \tag{17}$$

The derived equilibrium coincides with a *Pareto optimal risk exchange* (Borch (1962, 1968)).

Bühlmann (1984) extended his economic premium principle allowing general utility functions for market participants. He derived that under the weak assumption of concavity of utility functions, equilibrium prices are locally equal to the equilibrium prices derived under the assumption of exponential utility functions. Globally, equilibrium prices are similar, the only difference being that the absolute coefficient of risk aversion is no longer constant but depends on the agent's individual wealth. The interested reader is referred to Iwaki, Kijima & Morimoto (2001) for an extension of the above model to a multi-period setting. Connections between (15) and (6) are studied in Wang (2003).

2.3 Arbitrage-free Pricing

In the past decade, we have seen the emergence of a range of financial instruments with underlying risks that are traditionally considered to be "insurance risks". Examples are catastrophe derivatives and weather derivatives. Such developments on the interface between insurance and finance have put forward questions on appropriate pricing principles for these instruments.

As explained above, traditional actuarial pricing of insurance relies on economic theories of decision under uncertainty. Financial pricing of contingent claims mainly relies on risk-neutral valuation and applies an equivalent martingale measure as a risk-adjusted operator; see among many others Duffie (1996), Bingham & Kiesel (2004), Björk (2004) and Protter (2004) for textbook treatments. Risk-neutral valuation can be justified within a no-arbitrage setting. For the original work on the relation between the condition of no arbitrage and the existence of an equivalent martingale measure, we refer to Harrison & Kreps (1979) and Harrison & Pliska (1981). The basic idea of valuation by adjusting the primary asset process is from Cox & Ross (1976). As is well-known, arbitrage-free pricing may well fit in an equilibrium pricing framework; see e.g., Iwaki, Kijima & Morimoto (2001) in an insurance context.

While the no-arbitrage assumption is questionable for traditional insurance markets (see a.o. the discussion in Venter (1991, 1992) and Albrecht (1992)), one may argue that it does hold for insurance products that are traded on the financial markets (securitized insurance risks). Also, especially in life insurance a significant part of the risk borne is financial risk (interest rate risk, equity risk) rather than pure insurance risk; see e.g., Brennan & Schwartz (1976) and Pelsser & Laeven (2006). Arbitrage-free valuation is the natural device for the valuation of the financial portion of the insurance risk portfolio. Below we focus attention on pure insurance risk.

Several problems arise in applying financial no-arbitrage pricing methodology to securitized insurance risks. The principal problem is that of market incompleteness when the underlying assets are not traded. Indeed, most securitized insurance risks are *index-based* rather than (traded) asset-based. Market incompleteness implies that the contingent claim process cannot be hedged and therefore no-arbitrage arguments do in general not supply a unique equivalent martingale measure for risk-neutral valuation. The problem of market incompleteness proves to be even more serious when insurance processes are assumed to be stochastic jump processes. In the incomplete financial market there exist an infinite number of equivalent martingale measures and hence an infinite collection of prices.

The Esscher transform has proven to be a natural candidate and a valuable tool for the pricing of insurance and financial products. In their seminal work, Gerber & Shiu (1994, 1996) show that the Esscher transform can be employed to price derivative securities if the logarithms of the underlying asset process follow a stochastic process with stationary and independent increments (Lévy processes). Whereas the Esscher transform of the corresponding actuarial premium principle establishes a change of measure for a random variable, here the Esscher transform is defined more generally as a change of measure for certain stochastic processes. The idea is to choose the Esscher parameter (or in the case of multi assets: parameter vector) such that the discounted price process of each underlying asset becomes a martingale under the Esscher transformed probability measure. In the case where the equivalent martingale measure is unique it is obtained by the Esscher transform. In the case where the equivalent martingale measure is not unique (the usual case in insurance), the Esscher transformed probability measure can be justified if there is a representative agent maximizing his expected utility with respect to a power utility function.

Inspired by this, Bühlmann *et al.* (1996) more generally use *conditional Esscher transforms* to construct equivalent martingale measures for classes of semi-martingales; see also Kalssen & Shyriaev (2002) and Jacod & Shyriaev (2003).

In Goovaerts & Laeven (2007), the approach of establishing risk evaluation mechanisms by means of an axiomatic characterization is used to characterize a pricing mechanism that can generate approximate-arbitrage-free financial derivative prices. The price representation derived, involves a probability measure transform that is closely related to the Esscher transform; it is called the *Esscher-Girsanov transform.* In a financial market in which the primary asset price is represented by a general stochastic differential equation, the price mechanism based on the Esscher-Girsanov transform can generate approximate-arbitrage-free financial derivative prices.

Below we briefly outline the simple model of Gerber & Shiu (1994, 1996) for a single primary risk. Assume that there is a stochastic process $\{X(t)\}_{t\geq 0}$ with independent and stationary increments and $X(0) = 0$, such that

$$S(t) = S(0)e^{X(t)}, \qquad t \geq 0. \tag{18}$$

Assume furthermore that the moment generating function of $X(t)$, denoted by $m_X(\alpha, t)$ exists for all $\alpha > 0$ and all $t > 0$. Notice that the stochastic process

$$\left\{e^{\alpha X(t)} m_X(\alpha, 1)^{-t}\right\}_{t\geq 0}$$

is a positive martingale and can be used to establish a change of probability measure. Gerber & Shiu (1994, 1996) then define the risk-neutral Esscher measure of parameter α^* such that the process

$$\left\{e^{-rt} S(t)\right\}_{t\geq 0}$$

is a martingale with respect to the Esscher measure with parameter α^*; here r denotes the deterministic risk-free rate of interest. Let us denote the Esscher measure with parameter α by $\mathbb{Q}(\alpha)$ and the true probability measure by $\mathbb{P}$. From the martingale condition

$$\begin{aligned} S(0) &= \mathbb{E}^{\mathbb{Q}(\alpha^*)}\left[e^{-rt}S(t)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[e^{-rt}e^{\alpha^* X(t)}m_X(\alpha^*,1)^{-t}S(t)\right], \end{aligned}$$

we obtain

$$\begin{aligned} e^{rt} &= \mathbb{E}^{\mathbb{P}}\left[e^{(1+\alpha^*)X(t)}m_X(\alpha^*,1)^{-t}\right] \\ &= \left(\frac{m_X\left(1+\alpha^*,1\right)}{m_X\left(\alpha^*,1\right)}\right)^t \end{aligned}$$

or equivalently

$$\mathbb{E}\left[e^{(\alpha^*+1)X(1)}\right] = \mathbb{E}\left[e^{\alpha^* X(1)+r}\right].$$

Notice that the parameter α^* is unique since

$$\frac{m_X\left(1+\alpha,1\right)}{m_X\left(\alpha,1\right)},$$

is strictly increasing in α.

A martingale approach to premium calculation for the special case of compound Poisson processes, the classical risk process in insurance, was studied by Delbaen & Haezendonck (1989).

3 Premium Principles and Their Properties

Many of the (families of) premium principles that emerge from one of the theories discussed in Section 2 can be (directly) characterized axiomatically. The general purpose of an axiomatic characterization is to demonstrate what are the essential assumptions to be imposed and what are relevant parameters or concepts to be determined. A premium principle is appropriate if and only if its characterizing axioms are. Axiomatizations can be used to justify a premium principle, but also to criticize it.

A systematic study of the properties of premium calculation principles and their axiomatic characterizations was pioneered by Goovaerts, De Vylder & Haezendonck (1984). Below we list various properties that premium principles may (or may not) satisfy.

3.1 Properties of Premium Principles

Definition 9 (Law invariance (independence, objectivity)) *π is ($\mathbb{P}$-)law invariant if $\pi[X]=\pi[Y]$ when $\mathbb{P}[X\le x]=\mathbb{P}[Y\le x]$ for all real x.*

$\mathbb{P}$-law invariance means that for a given risk X the premium $\pi[X]$ depends on X only via the distribution function $\mathbb{P}[X\le x]$. We note that, to have equal ($\mathbb{P}$)-distributions the random variables X and Y need in general not be defined on the same measurable space.

Definition 10 (Monotonicity) *π is monotonic if $\pi[X] \leq \pi[Y]$ when $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. π is $\mathbb{P}$-monotonic if $\pi[X] \leq \pi[Y]$ when $X \leq Y$ $\mathbb{P}$-almost surely.*

Definition 11 (Preserving first-order stochastic dominance (FOSD)) *π preserves first-order stochastic dominance if $\pi[X] \leq \pi[Y]$ when $\mathbb{P}[X \leq x] \geq \mathbb{P}[Y \leq x]$ for all $x \in \mathbb{R}$.*

Law invariance together with monotonicity implies preserving FOSD.

Definition 12 (Preserving stop-loss order (SL)) *π preserves stop-loss order if $\pi[X] \leq \pi[Y]$ when $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \in \mathbb{R}$.*

We note that preserving SL implies preserving FOSD.

It is well-known that if the utility function is non-decreasing, which is implied by the monotonicity axiom that is imposed to axiomatize expected utility, then $X \leq_{\text{FOSD}} Y$ implies ordered equivalent utility premiums. If moreover the expected utility maximizer is risk-averse, which is equivalent to concavity of the utility function, then $X \leq_{\text{SL}} Y$ implies ordered equivalent utility premiums; see Hadar & Russell (1969) and Rothschild & Stiglitz (1970) for seminal work on the ordering of risks within the framework of expected utility theory. For an extensive account on stochastic ordering in actuarial science, see Denuit *et al.* (2005).

Definition 13 (Risk loading) *π induces a risk loading if $\pi[X] \geq \mathbb{E}[X]$.*

Definition 14 (Not unjustified) *π is not unjustified if $\pi[c] = c$ for all real c.*

Definition 15 (No ripoff (Non-excessive loading)) $\pi[X] \leq \operatorname{esssup}[X]$.

Notice that $\mathbb{P}$-monotonicity implies no ripoff.

Definition 16 (Additivity) *π is additive if $\pi[X+Y] = \pi[X] + \pi[Y]$.*

The assumption of no-arbitrage implies that the pricing rule is additive.

Definition 17 (Translation invariance (tr. equivariance, translativity, consistency)) *π is translation invariant if $\pi[X+c] = \pi[X] + c$ for all real c.*

Definition 18 (Positive homogeneity (scale invariance, scale equivariance)) *π is positively homogeneous if $\pi[aX] = a\pi[X]$ for all $a \geq 0$.*

Definition 19 (Subadditivity resp. Superadditivity) *π is subadditive (resp. superadditive) if $\pi[X+Y] \leq \pi[X] + \pi[Y]$ (resp. $\pi[X+Y] \geq \pi[X] + \pi[Y]$)*

Definition 20 (Convexity) *π is convex if $\pi[\alpha X + (1-\alpha)Y] \leq \alpha\pi[X] + (1-\alpha)\pi[Y]$ for all $\alpha \in (0,1)$.*

Notice that under positive homogeneity, subadditivity and convexity are equivalent. See Deprez & Gerber (1985) for an early account of convex premium principles.

Definition 21 (Independent additivity) *π is independent additive if $\pi[X + Y] = \pi[X] + \pi[Y]$ when X and Y are independent.*

See Gerber (1974) and Goovaerts *et al.* (2004b) for axiomatizations of independent additive premium principles.

Definition 22 (Comonotonic additivity) *π is comonotonic additive if $\pi[X + Y] = \pi[X] + \pi[Y]$ when X and Y are comonotonic.*

Definition 23 (Iterativity) *π is iterative if $\pi[X] = \pi[\pi[X|Y]]$.*

In addition to the above-mentioned properties, axiomatic characterizations of premium principles often impose technical (mainly continuity) conditions, required for obtaining mathematical proofs. Such conditions appear in various forms (e.g., if X_n converges weakly to X then $\pi[X_n] \to \pi[X]$) and are usually difficult to interpret (justify) economically.

3.2 A Plethora of Premium Principles

In this section, we list many well-known premium principles and tabulate which of the properties from the Section "Properties of Premium Principles" above they satisfy. The theories presented in Section 2 are not mutually exclusive. Some premium principles (Esscher) arise from more than one theory. Other premium principles (Dutch) instead are not directly based on any of the above-mentioned theories nor on an axiomatic characterization, but rather on the nice properties that they exhibit.

Definition 24 (Expected value principle) *The expected value principle is given by*

$$\pi[X] = (1 + \lambda)\mathbb{E}[X], \qquad \lambda \geq 0. \tag{19}$$

If $\lambda = 0$, the net premium is obtained. The net premium can be justified as follows: for the risk $X := \sum_{i=1}^{n} X_i$ of a large homogeneous portfolio, with $X_i \sim$ iid and $\mathbb{E}[X_i] = \mu < \infty$, we have on behalf of the law of large numbers, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[|X - n\mu| < n\varepsilon\right] = 1. \tag{20}$$

Hence, in case the net premium μ is charged for each risk transfer X_i, the premium income μ is sufficient to cover the average claim size $\frac{X}{n}$ with probability 1, as n tends to ∞.

However, in the expected utility framework the net premium suffices only for a risk-neutral insurer. Moreover, it follows from ruin theory that if no loading is applied, ruin will occur with certainty. In this spirit, the loading factor λ can be determined by setting sufficiently protective solvency margins. Such solvency margins may be derived from ruin estimates of the underlying risk process over a given period of time; see e.g., Gerber (1979) or Kaas *et al.* (2001). This applies in a similar way to some of the premium principles discussed below. For $\lambda > 0$, the loading margin increases with the expected value.

Definition 25 (Equivalent utility (zero utility) principle) *See Section 2.1.*

Definition 26 (Distortion principle) *See Section 2.1.*

Definition 27 (Mean value principle) *For a given non-decreasing and non-negative function f on $\mathbb{R}$ the mean value principle is the root of*

$$f(\pi) = \mathbb{E}[f(X)]. \tag{21}$$

Definition 28 (Variance Principle) *The variance principle is given by*

$$\pi[X] = \mathbb{E}[X] + \lambda \mathrm{Var}[X], \qquad \lambda > 0. \tag{22}$$

Definition 29 (Standard Deviation Principle) *The standard deviation principle is given by*

$$\pi[X] = \mathbb{E}[X] + \lambda \sqrt{\mathrm{Var}[X]}, \qquad \lambda > 0. \tag{23}$$

See Denneberg (1990) for critical comments on the standard deviation principle, advocating the use of the absolute deviation rather than the standard deviation. Dynamic versions of the variance and standard deviation principles in an economic environment are studied by Schweizer (2001) and Møller (2001).

Definition 30 (Exponential Principle) *The exponential principle is given by*

$$\pi[X] = \frac{1}{\alpha} \log \mathbb{E}[\exp(\alpha X)], \qquad \alpha > 0. \tag{24}$$

See Gerber (1974), Bühlmann (1985), Goovaerts *et al.* (2004b) and Young & Zariphopoulou (2004) for axiomatic characterizations and applications of the exponential principle.

Consider a random variable X and a real number $\alpha > 0$ for which $\mathbb{E}[e^{\alpha X}]$ exists. The positive random variable

$$\frac{e^{\alpha X}}{\mathbb{E}[e^{\alpha X}]} \tag{25}$$

with expectation equal to one, can be used as a Radon-Nikodym derivative to establish a change of probability measure. The thus obtained probability measure is called the *Esscher* measure and is *equivalent* to the original probability measure (i.e., the probability measures are mutually absolutely continuous). The Esscher transform was first introduced by the Swedish actuary F. Esscher, see Esscher (1932).

Definition 31 (Esscher Principle) *The Esscher premium is given by*

$$\pi[X] = \frac{\mathbb{E}\left[X e^{\alpha X}\right]}{\mathbb{E}\left[e^{\alpha X}\right]}, \qquad \alpha > 0. \tag{26}$$

As an alternative representation, consider the cumulant generating function (cgf) of a random variable X:

$$\kappa_X(\alpha) := \log\left(\mathbb{E}\left[e^{\alpha X}\right]\right). \tag{27}$$

The first and second order derivatives of the cgf with respect to α are

$$\kappa'_X(\alpha) = \frac{\mathbb{E}\left[X e^{\alpha X}\right]}{\mathbb{E}\left[e^{\alpha X}\right]} \tag{28}$$

and

$$\kappa''_X(\alpha) = \frac{\mathbb{E}\left[X^2 e^{\alpha X}\right]}{\mathbb{E}\left[e^{\alpha X}\right]} - \left(\frac{\mathbb{E}\left[X e^{\alpha X}\right]}{\mathbb{E}\left[e^{\alpha X}\right]}\right)^2. \tag{29}$$

One can interpret $\kappa'_X(\alpha)$ (or the Esscher premium) and $\kappa''_X(\alpha)$ as the mean and the variance of the random variable X under the Esscher transformed probability measure.

Gerber & Goovaerts (1981) establish an axiomatic characterization of an additive premium principle that involves a mixture of Esscher transforms. A drawback of both the mixed and non-mixed Esscher premium is that it is not monotonic; see Gerber (1981) and Van Heerwaarden, Kaas & Goovaerts (1989). Goovaerts *et al.* (2004b) establish an axiomatic characterization of risk measures that are additive for independent random variables including an axiom that guarantees monotonicity. The premium principle obtained is a mixture of exponential premiums.

Definition 32 (Swiss principle) *For a given non-negative and non-decreasing function w on $\mathbb{R}$ and a given parameter $0 \leq p \leq 1$ the Swiss premium is the root of*

$$\mathbb{E}[w(X - p\pi)] = w((1-p)\pi). \tag{30}$$

Notice that the Swiss principle includes both the equivalent utility principle and the mean value principle as special cases; see Gerber (1974), Bühlmann *et al.* (1977) and Goovaerts, De Vylder & Haezendonck (1984).

Definition 33 (Dutch principle) *The Dutch principle is given by*

$$\pi[X] = \mathbb{E}[X] + \theta\mathbb{E}[(X - \alpha\mathbb{E}[X])_+], \qquad \alpha \geq 1, \quad 0 < \theta \leq 1. \tag{31}$$

The Dutch premium principle was introduced by Van Heerwaarden & Kaas (1992).

Definition 34 (Orlicz principle) *Let $X \geq 0$. For a given normalized Young function ψ on $\mathbb{R}_+ \cup \{0\}$ the Orlicz premium is the root of*

$$\mathbb{E}[\psi(X/\pi)] = 1. \tag{32}$$

The Orlicz principle is a multiplicative alternative to the equivalent utility principle; see Haezendonck & Goovaerts (1982).

All premium principles mentioned below are law invariant. All principles are not unjustified except for the expected value principle with $\lambda > 0$.

Table 1: Properties of Premium Principles

Principle	M	TI	PH	A	SA	C	IA	CA	SL	NR	RL	I
Expected value	+	− (+ for $\lambda = 0$)	+	+	+	+	+	+	+	− (+ for $\lambda = 0$)	+	− (+ for $\lambda = 0$)
Equivalent utility	+	+	−	−	−	+ (if u concave)	−	−	+ (if u concave)	+	+ (if u concave)	−
Distortion	+	+	+	−	+ (if $\bar{g}$ concave)	+ (if $\bar{g}$ concave)	−	+	+ (if $\bar{g}$ concave)	+	+ (if $\bar{g}(x) \geq x$)	−
Mean value	+	−	−	−	−	−	−	−	+ (if f convex)	+	+ (if f convex)	+
Variance	−	+	−	−	−	+	+	−	−	−	+	−
Standard deviation	−	+	+	−	+	+	−	−	−	−	+	−
Exponential	+	+	−	−	−	+	+	−	+	+	+	+
Esscher	−	+	−	−	−	−	+	−	−	+	+	−
Swiss	+	−	−	−	−	−	−	−	+ (if w convex)	+	+ (if w convex)	−
Dutch	+	−	+	−	+	+	−	−	+	+	+	−
Orlicz	+	−*	+	−	+	+	−	−	+	+	+	−

M = monotonicity; TI = translation invariance; PH = positive homogeneity; A = additivity; SA = subadditivity; C = convexity; IA = independent additivity; CA = comonotonic additivity; SL = stop-loss order; NR = no ripoff; RL = non-negative risk loading; I = iterativity.

* The so-called *Haezendonck risk measure*, which is an extension of the Orlicz premium (see Goovaerts *et al.* (2004a) and Bellini & Rosazza Gianin (2006)), is translation invariant.

4 A Generalized Markov Inequality

Many of the premium principles discussed above can be derived in a unified approach by minimizing a generalized Markov inequality; see Goovaerts *et al.* (2003). This approach involves two exogenous functions $v(\cdot)$ and $\phi(\cdot,\cdot)$ and an exogenous parameter $\alpha \leq 1$. Assuming that v is non-negative and non-decreasing and that ϕ satisfies $\phi(x,\pi) \geq 1_{\{x>\pi\}}$ one easily proves the following generalized Markov inequality:

$$\mathbb{P}\left[X > \pi\right] \leq \frac{\mathbb{E}\left[\phi(X,\pi)v(X)\right]}{\mathbb{E}[v(X)]}. \tag{33}$$

Below we consider the case where the upper bound in the above inequality reaches a given confidence level $\alpha \leq 1$:

$$\mathbb{P}\left[X > \pi\right] \leq \frac{\mathbb{E}\left[\phi(X,\pi)v(X)\right]}{\mathbb{E}[v(X)]} = \alpha \leq 1. \tag{34}$$

When $\alpha = 1$, the equation

$$\frac{\mathbb{E}\left[\phi(X,\pi)v(X)\right]}{\mathbb{E}[v(X)]} = 1, \tag{35}$$

can generate many well-known premium principles:

Table 2: Premium Principles Derived from Equation (35)

Premium principle	$v(x)$	$\phi(x,\pi)$
Mean value	1	$\frac{f(x)}{f(\pi)}$
Zero utility	1	$\frac{u(\pi-x)}{u(0)}$
Swiss	1	$\frac{w(x-p\pi)}{w((1-p)\pi)}$
Orlicz	1	$\psi(x/\pi)$

For all ϕ, v, by (34),

$$\pi \geq \inf\{x \in \mathbb{R} : \mathbb{P}[X \leq x] \geq 1-\alpha\},$$

which sheds light on the relation between premium principles and quantile-based solvency margins.

Acknowledgements We are grateful to Yichun Chi for indicating a typo and a minor inconsistency in Table 1, in April 2011, and to Myroslav Drozdenko for indicating a typo in the justification of the expected value principle, in July 2011. Roger Laeven acknowledges the financial support of the Netherlands Organization for Scientific Research (NWO Grant No. 42511013 and NWO VENI Grant 2006). Marc Goovaerts acknowledges the financial support of the GOA/02 Grant ("Actuariële, financiële en statistische aspecten van afhankelijkheden in verzekerings- en financiële portefeuilles").

References

[1] Albrecht, P. (1992). "Premium calculation without arbitrage? A note on a contribution by G. Venter," *Astin Bulletin* 22, 247-254.

[2] Allais, Maurice (1953). "Le comportement de l'homme rationnel devant le risque: critique des postulates et axiomes de l'école Américaine," *Econometrica* 21, 503-546.

[3] Bellini, Fabio & Emanuela Rosazza Gianin (2006). "On Haezendonck risk measures," Working Paper Univ. di Milano Bicocca.

[4] Bingham, Nick H. & Rüdiger Kiesel (2004). *Risk-Neutral Valuation*, 2nd edn., Berlin: Springer.

[5] Björk, Tomas (2004). *Arbitrage Theory in Continuous Time*, 2nd edn., Oxford: Oxford University Press.

[6] Borch, Karl H. (1962). "Equilibrium in a reinsurance market," *Econometrica* 3, 424-444.

[7] Borch, Karl H. (1968). *The Economics of Uncertainty*, Princeton: Princeton University Press.

[8] Borch, Karl H. (1974). *The Mathematical Theory of Insurance*, Toronto: Lexington Books.

[9] Brennan, Michael J. & Eduardo S. Schwartz (1976). "The pricing of equity-linked life insurance policies with an asset value guarantee," *Journal of Financial Economics* 3, 195-213.

[10] Bühlmann, Hans (1970). *Mathematical Methods in Risk Theory*, Berlin: Springer Verlag.

[11] Bühlmann, Hans, B. Gagliardi, Hans U. Gerber & E. Straub (1977). "Some inequalities for stop-loss premiums," *Astin Bulletin* 9, 75-83.

[12] Bühlmann, Hans (1980). "An economic premium principle," *Astin Bulletin* 11, 52-60.

[13] Bühlmann, Hans (1984). "The general economic premium principle," *Astin Bulletin* 14, 13-22.

[14] Bühlmann, Hans (1985). "Premium calculation from top down," *Astin Bulletin* 15, 89-101.

[15] Bühlmann, Hans, Freddy Delbaen, Paul Embrechts & Albert N. Shiryaev (1996). "No-arbitrage, change of measure and conditional Esscher transforms," *CWI Quarterly* 9, 291-317.

[16] Choquet, Gustave (1953-4). "Theory of capacities," *Annales de l'Institut Fourier* 5 (Grenoble), 131-295.

[17] Cox, John & Stephen Ross (1976). "The valuation of options for alternative stochastic processes," *Journal of Financial Economics* 3, 145-166.

[18] Delbaen, Freddy & Jean Haezendonck (1989). "A martingale approach to premium calculation principles in an arbitrage free market," *Insurance: Mathematics and Economics* 8, 269-277.

[19] Denneberg, Dieter (1990). "Premium calculation: Why standard deviation should be replaced by absolute deviation," *Astin Bulletin* 20, 181-190.

[20] Denneberg, Dieter (1994). *Non-additive Measure and Integral*, Boston: Kluwer Academic Publishers.

[21] Denuit, Michel, Jan Dhaene, Marc J. Goovaerts & Rob Kaas (2005). *Actuarial Theory for Dependent Risks*, New York: Wiley.

[22] Denuit, Michel, Jan Dhaene, Marc J. Goovaerts, Rob Kaas & Roger J.A. Laeven (2006). "Risk measurement with equivalent utility principles," In: Rüschendorf, Ludger (Ed.), *Risk Measures: General Aspects and Applications* (special issue), *Statistics and Decisions* 24, 1-26.

[23] Deprez, Olivier & Hans U. Gerber (1985), "On convex principles of premium calculation," *Insurance: Mathematics and Economics* 4, 179-189.

[24] Duffie, Darrell (1996). *Dynamic Asset Pricing Theory*, 2nd edn., Princeton: Princeton University Press.

[25] Embrechts, Paul (2000). "Actuarial versus financial pricing of insurance," *Risk Finance* 1, 17-26.

[26] Esscher, Fredrik (1932). "On the probability function in the collective theory of risk," *Scandinavian Actuarial Journal* 15, 175-195.

[27] Föllmer, Hans & Alexander Schied (2004). *Stochastic Finance*, 2nd Edition, Berlin: De Gruyter.

[28] Gerber, Hans U. (1974). "On additive premium calculation principles," *Astin Bulletin* 7, 215-222.

[29] Gerber, Hans U. (1979). *An Introduction to Mathematical Risk Theory*, Huebner Foundation Monograph 8, Illinois: Homewood.

[30] Gerber, Hans U. & Marc J. Goovaerts (1981). "On the representation of additive principles of premium calculation," *Scandinavian Actuarial Journal* 4, 221-227.

[31] Gerber, Hans U. (1981). "The Esscher premium principle: A criticism, Comment," *Astin Bulletin* 12, 139-140.

[32] Gerber, Hans U. & Elias S.W. Shiu (1994). "Option pricing by Esscher transforms," *Transactions of the Society of Actuaries* 46, 99-191.

[33] Gerber, Hans U. & Elias S.W. Shiu (1996). "Actuarial bridges to dynamic hedging and option pricing," *Insurance: Mathematics and Economics* 18, 183-218.

[34] Gilboa, Itzhak & David Schmeidler (1989). "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics* 18, 141-153.

[35] Goovaerts, Marc J., F. Etienne C. De Vylder & Jean Haezendonck (1984). *Insurance Premiums*, Amsterdam: North Holland Publishing.

[36] Goovaerts, Marc J., Rob Kaas, Jan Dhaene & Qihe Tang (2003). "A unified approach to generate risk measures," *Astin Bulletin* 33, 173-191.

[37] Goovaerts, Marc J., Rob Kaas, Jan Dhaene & Qihe Tang (2004a). "Some new classes of consistent risk measures," *Insurance: Mathematics and Economics* 34, 505-516.

[38] Goovaerts, Marc J., Rob Kaas, Roger J.A. Laeven & Qihe Tang (2004b). "A comonotonic image of independence for additive risk measures," *Insurance: Mathematics and Economics* 35, 581-594.

[39] Goovaerts, Marc J. & Roger J.A. Laeven (2007). "Actuarial risk measures for financial derivative pricing," *Insurance: Mathematics and Economics*, in press.

[40] Greco, Gabriele (1982). "Sulla Rappresentazione di Funzionali Mediante Integrali," Rend. Sem. Mat. Univ. Padova 66, 21-42.

[41] Hadar, Joseph & William R. Russell (1969). "Rules for ordering uncertain prospects," *American Economic Review* 59, 25-34.

[42] Haezendonck, Jean & Marc J. Goovaerts (1982). "A new premium calculation principle based on Orlicz norms," *Insurance: Mathematics and Economics* 1, 41-53.

[43] Harrison, J. Michael & David M. Kreps (1979). "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory* 20, 381-408.

[44] Harrison, J. Michael & Stanley R. Pliska (1981). "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and Their Applications* 11, 215-260.

[45] Iwaki, Hideki, Masaaki Kijima & Yuji Morimoto (2001). "An economic premium principle in a multiperiod economy," *Insurance: Mathematics and Economics* 28, 325-339.

[46] Jacod, Jean & Albert N. Shiryaev (2003). *Limit Theorems for Stochastic Processes*, 2nd Edition, Section III.7.b, Springer, New York.

[47] Kaas, Rob, Angela E. Van Heerwaarden, & Marc J. Goovaerts (1994). *Ordering of Actuarial Risks*, Education Series 1, CAIRE, Brussels.

[48] Kaas, Rob, Marc J. Goovaerts, Jan Dhaene & Michel Denuit (2001). *Modern actuarial risk theory*, Dordrecht: Kluwer Academic Publishers.

[49] Kallsen, Jan & Albert N. Shiryaev (2002). "The cumulant process and Esscher's change of measure," *Finance and Stochastics* 6, 397-428.

[50] Karatzas, Ioannis & Steven E. Shreve (1988). *Brownian Motion and Stochastic Calculus*, New York: Springer.

[51] Laeven, Roger J.A. (2005). *Essays on Risk Measures and Stochastic Dependence*, Tinbergen Institute Research Series 360, Chapter 3, Thela Thesis, Amsterdam.

[52] Møller, Thomas (2001). "On transformations of actuarial valuation principles," *Insurance: Mathematics and Economics* 28, 281-303.

[53] Pelsser, Antoon A.J. & Roger J.A. Laeven (2006). "Optimal dividends and ALM under unhedgeable risk," Working Paper University of Amsterdam.

[54] Protter, Philip (2004). *Stochastic Integration and Differential Equations*, 2nd edn., Berlin: Springer.

[55] Quiggin, John (1982). "A theory of anticipated utility," *Journal of Economic Behavior and Organization* 3, 323-343.

[56] Rothschild, Michael & Joseph E. Stiglitz (1970). "Increasing risk I: A definition," *Journal of Economic Theory* 2, 225-243.

[57] Savage, Leonard J. (1954). *The Foundations of Statistics*, New York: Wiley.

[58] Schmeidler, David (1986). "Integral representation without additivity," *Proceedings of the American Mathematical Society* 97, 255-261.

[59] Schmeidler, David (1989). "Subjective probability and expected utility without additivity," *Econometrica* 57, 571-587.

[60] Schweizer, Martin (2001). "From actuarial to financial valuation principles," *Insurance: Mathematics and Economics* 28, 31-47.

[61] Sugden, Robert (1997). "Alternatives to expected utility theory: Foundations and concepts," In: *Handbook of Expected Utility Theory*, Eds. S. Barberà, P.J. Hammond & C. Seidl, Boston: Kluwer.

[62] Van Heerwaarden, Angela E., Rob Kaas & Marc J. Goovaerts (1989). "Properties of the Esscher premium calculation principle," *Insurance: Mathematics and Economics* 8, 261-267.

[63] Van Heerwaarden, Angela E. & Rob Kaas (1992). "The Dutch premium principle," *Insurance: Mathematics and Economics* 11, 129-133.

[64] Venter, Gary G. (1991). "Premium calculation implications of reinsurance without arbitrage," *Astin Bulletin* 21, 223-230.

[65] Venter, Gary G. (1992). "Premium calculation without arbitrage Authors reply on the note by P. Albrecht," *Astin Bulletin* 22, 255-256.

[66] Von Neumann, John & Oscar Morgenstern (1947). *Theory of Games and Economic Behavior*, 2nd edn., Princeton: Princeton University Press.

[67] Wang, Shaun S. (1996). "Premium calculation by transforming the layer premium density," *Astin Bulletin* 26, 71-92.

[68] Wang, Shaun S., Virginia R. Young, & Harry H. Panjer (1997). "Axiomatic characterization of insurance prices," *Insurance: Mathematics and Economics* 21, 173-183.

[69] Wang, Shaun S. (2003). "Equilibruim pricing transforms: new results using Bühlmann's 1980 economic model," *Astin Bulletin* 33, 57-73.

[70] Yaari, Menahem E. (1987). "The dual theory of choice under risk," *Econometrica* 55, 95-115.

[71] Young, Virginia R. & Thaleia Zariphopoulou (2002). "Pricing dynamic insurance risks using the principle of equivalent utility," *Scandinavian Actuarial Journal* 2002, 246-279.

Roger J.A. Laeven
Tilburg University and CentER
Dept. of Econometrics and Operations Research
P.O. Box 90153
5000 LE Tilburg, The Netherlands

Marc J. Goovaerts
Catholic University of Leuven
Dept. of Applied Economics
Naamsestraat 69
B-3000 Leuven, Belgium

and

University of Amsterdam
Dept. of Quantitative Economics
Roetersstraat 11
1018 WB Amsterdam, The Netherlands