Stress-testing the Impact of Group Dependence on Credit Portfolio Risk

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Abstract

The default risk of firms is driven by firm-specific factors but also by systematic factors and the latter are responsible for default dependence between different firms. Another source of default dependence is structural links between firms. For example, a mother company may consist of different legal entities and a default of the former may be contagious and lead to the default of all others, i.e. strong dependence is present in this case. Conversely a possible default from one of the constituent companies may be prevented by the mother company. In fact, such dependence or guarantee considerations are often made when assessing the individual default probabilities, and then typically result in assigning lower default probabilities to daughter companies.

While it is correct to consider these direct dependence relations when assessing the single default probabilities they also need to be considered when modelling the aggregate loss but it appears that this is ignored by the current state-of-the-art credit risk portfolio models. In this paper we will use the CreditRisk$^+$ model to stress-test the direct (intra-)group dependences or contagion effects by making these as strong as possible while leaving the other characteristics of the portfolio unchanged. Then, we show how this model can still be readily applied without major modifications. We also show that the CreditRisk$^+$ model will allow us to derive the loss distribution function explicitly.

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1 Introduction

For the last decade the financial industry has put models in place to assess the default risk of their various credit portfolios and Koyluogly & Hickman (1998) have classified these as follows:

1. the "Merton-based" approach; see J.P. Morgan's CreditMetrics (Gupton *et al.* (1997)) or MKMV's PortfolioManager (Zeng & Zhang (2001)),
2. the "Econometric" approach; see McKinsey & Company's CreditPortfolioView (Wilson (1997a,b),
3. the "Actuarial" approach; see CreditRisk$^+$ (Credit Suisse Financial Products (1997)).

While the mathematical properties of these different approaches are now well understood it remains difficult to prove the accuracy of either model, especially when measuring upper tails of the portfolio loss distribution. This is essentially because the portfolio will be most severely hit when several credit exposures default together. Unfortunately, since a default is a rare event it is difficult to predict default probabilities and even more so for joint default probabilities. At best financial institutions have a good view on the single and pairwise default probabilities, or equivalently, the default correlations but not on the likelihood that three or more loans default together. Then, as shown by Embrechts *et al.* (2003) and Frey *et al.* (2001) it is not difficult to build credit risk models that are consistent with this maximum available information while providing very different results. In this context, we believe that for credit portfolio calculations it is advisable to stick to a transparent - but still consistent - portfolio model which incorporates the essential features of credit risk.

Moreover it becomes important to stress-test the chosen model under various scenarios. First, this will reveal the sensitivity of the results to changing assumptions and parameters, and as such shed more light on the robustness of the model as well as the extent to which the portfolio may be impacted by adverse events. Next, the results can be used to steer discussion of the appropriate risk appetite and the amount of capital that should be set aside to cover the credit portfolio risk; see also Rösch and Scheule (2007) for more information.

In order to model credit portfolio risk we need to model the marginal risks as well as the way they interact, i.e. the dependencies. Dependence modelling in a credit context usually focuses on finding the economic dimensions that influence the default behaviour for the different loans. Apart from factors describing the global state of the economy such as, for example, interest rates the default drivers that are typically considered are asset size, industry sector and geographical situation; see Lopez (2002), Duellmann and Scheule (2003) or Dietsch and Petey (2004). As a result companies of similar size, industry activity and geographical situation will be grouped together meaning that they behave similarly, which is akin to saying that they are positively dependent.

However, all these dimensions together still do not fully capture all sources of dependence and in this paper we will assess the potential impact for some of these sources of hidden (or missed) dependence across portfolios. Indeed, a credit risk portfolio often contains groups of individual risks that are directly connected to each other.

As a first example let us mention the banking sector where the myriad of credit relations that exist between the different players is creating strong default dependence. The stand alone insolvency of a big bank will have a negative effect on the value of outstanding interbank claims and may as such lead to a series of other defaults. Secondly, mother companies may contain different legal entities and a default of the former may be contagious and lead to the default of all others, i.e. strong dependence is present in this case. Vice versa, a possible default from one of the constituent companies may be prevented by the mother company. In fact, such dependence or guarantee considerations are often made when assessing the individual default probabilities within a group of related companies, and then typically result in assigning lower default probabilities to the daughter companies.

While it is correct to consider these direct dependence relations when assessing the default probabilities of the different credit risks, they also need to be considered in the modelling of the aggregate loss distribution but it appears that in practice this is often ignored. Indeed the three mentioned classes of portfolio models all assume that conditional on the state of economic variables the different risks are independent, and as such ignore direct dependence.

In this paper we will stress-test the direct (intra-group) dependence by making it as strong as possible while leaving the other characteristics of the portfolio unchanged. We will use the standard version of the CreditRisk$^+$

model to do so. Specifically we show how the traditional CreditRisk$^+$ model can still be readily applied without major modifications to account for intra-group dependence and contagious risk. We will show that the CreditRisk$^+$ model will allow us to derive the relevant risk measures in closed form. We will also numerically illustrate that the impact of a stressed intra-group dependence is significant and should not be ignored by practitioners.

The main reasons for working with the CreditRisk$^+$ model are that very few assumptions are required and that its natural parameterisation involves default statistics only. We remark that confronted with a lack of sufficient directly observable default data, financial institutions and also software providers often resort to so-called *asset correlations*. In fact, the Portfolio Models that use the "Merton-based" approach such as MKMV's Portfolio Manager and J.P. Morgan's CreditMetrics have all been developed in this setting. These asset correlations can indeed be transformed into pairwise default probabilities (default correlations) and even all joint default probabilities by means of the so-called *Merton Model* of the firm; we refer the interested reader to Crouhy *et al.* (2000) for further details. Whilst the wider availability of asset data and the additional sophistication may provide some feeling of comfort, this comes at the cost of more model risk. Indeed it must be pointed out that the asset data is transformed into default values through a model which should then be compared with available default statistics. We agree with Frye (2008) however that many 'naïve' risk managers rely on asset correlations without checking the results. Frye (2008) also nicely shows that doing so may result in erroneous risk management practices; see also Chernih *et al.* (2006). Moreover asset data itself is not observable but needs to be derived from equity data using option models; see Crouhy (2000). To the best of our knowledge there is no indication that this more involved portfolio approach provides more reliable portfolio risk measures. For a study on how credit risk models can be parameterised using observed default data we refer to Hamerle & Rösch (2007). Note that Vandendorpe *et al*. (2008) have shown how the CreditRisk$^+$ model can be parameterised using asset correlations, hence building a bridge between the actuarial model and the Merton-based approach.

The structure of the paper is as follows: In Section 2, we introduce the CreditRisk$^+$ which in actuarial circles is known to be an example of a collective risk model. Next, in Section 3 we propose our extension to incorporate direct (intra-) group dependence and show how this still fits from a technical point of view into the CreditRisk$^+$ framework. Section 4 provides numerical illustrations and Section 5 concludes.

2 The standard CreditRisk$^+$ Model

2.1 Description

In CreditRisk$^+$ the distribution of the Portfolio Credit Loss is approximated using a so-called collective risk model. We will deal with this approximation in its most simple and transparent form; see also Dhaene *et al.* (2003) or Vandendorpe *et al.* (2008).

To this end let us consider a portfolio consisting of n credit risks and let for $i = 1, 2, ..., n$ the random variable I_i be defined as the indicator variable which equals 1 if risk i leads to failure in the next reference period (e.g., one year time horizon) and 0 otherwise. The default probability is stochastic and will be denoted by Q_i with mean $E[Q_i] = q_i$. We find that the probability that risk i leads to a failure in the corresponding period, is given by q_i:

$$\Pr[I_i = 1] = E[Q_i] = q_i. \tag{1}$$

Further for risk i, let $(EAD)_i$ denote the "Exposure-At-Default" and $(LGD)_i$ the "Loss-Given-Default" for the risk horizon under consideration. The "Exposure-At-Default" is the maximal amount of loss for risk i, given default occurs whereas the "Loss-Given-Default" is the percentage of the loss on policy i, given default occurs. The "Aggregate Credit Portfolio Loss" (the aggregate loss for short) during the reference period is then given by

$$S = \sum_{i=1}^{n} I_i \ (EAD)_i \ (LGD)_i \, . \tag{2}$$

We will assume that the $(EAD)_i$ and $(LGD)_i$ are deterministic. The results of these paper can be generalised to account for stochastic LGD and EAD but we will not deal with this additional complexity here. Note that the use (and abuse) of stochastic Loss-Given-Defaults is considered in Dhaene $et\ al$. (2005) whereas Bürgisser (2001) provides a generalisation of the CreditRisk$^+$ model that deals with stochastic dependent LGD's. Also Rösch and Scheule (2007) have studied a framework which includes the dependence between the different mentioned risk components.

Note that the random variable of interest S can be recasted as a sum of n compound Bernoulli random variables

$$S = \sum_{i=1}^{n} I_i \; b_i, \tag{3}$$

with

$$b_i = (EAD)_i \; (LGD)_i \,. \tag{4}$$

The aggregate loss S is the sum of the losses on the individual credit risks. In order to compute the distribution function of S, exact knowledge of the multivariate distribution function for $(I_1, I_2, ..., I_n)$ is principally required. However, since the dependent Bernoulli r.v's I_i are too difficult to work with, we will replace them by other dependent r.v's N_i that are "close" to the I_i but more tractable. Hence instead of considering the distribution function of the aggregate loss in the individual model,

$$S = \sum_{i=1}^{n} I_i \; b_i = \sum_{i=1}^{n} \sum_{j=1}^{I_i} \; b_i \qquad \text{with} \quad \Pr\left[I_i = 1\right] = q_i, \tag{5}$$

we will focus on the distribution function of the following random variable:

$$S^* = \sum_{i=1}^{n} \sum_{j=1}^{N_i} b_i. \tag{6}$$

In order to introduce the dependence caused by common factors, we will consider a "Bayesian approach". Therefore, let us assume that there exists a random variable Λ -representing the "global state of the economy"- such that, conditionally given $\Lambda = \lambda$, the random variables N_i are mutually independent:

$$(N_i \mid \Lambda = \lambda) \;\text{ are mutually independent.} \tag{7}$$

Furthermore, we will also assume that, conditionally given $\Lambda = \lambda$, the random variables N_i are Poisson distributed with parameters $q_i \; \lambda$:

$$(N_i \mid \Lambda = \lambda) \stackrel{d}{=} \text{Poisson}\left(q_i \; \lambda\right). \tag{8}$$

Note that the default intensities are stochastic since they depend on the random variable Λ and are in fact given by $q_i\Lambda$. Finally, we will take Λ

as a Gamma distributed random variable with parameters α and β. It is well-known that the moment generating function of N_i is now given by:

$$\begin{aligned} E\left[e^{tN_i}\right] &= E\left[E\left[e^{tN_i} \mid \Lambda\right]\right] \\ &= E\left[\exp\left(q_i\ \Lambda\ \left(e^t - 1\right)\right)\right] \\ &= m_\Lambda\left(q_i\ \left(e^t - 1\right)\right) \\ &= \left(\frac{\beta}{\beta - q_i\ \left(e^t - 1\right)}\right)^\alpha \\ &= \left(\frac{\frac{\beta}{\beta+q_i}}{1 - \left(1 - \frac{\beta}{\beta+q_i}\right) e^t}\right)^\alpha, \end{aligned} \tag{9}$$

which implies that

$$N_i \stackrel{d}{=} NB\left(\alpha, \frac{\beta}{\beta + q_i}\right), \tag{10}$$

i.e., the random variables N_i are negative binomial distributed but not independent. Under these assumptions, the random variable S^* defined by

$$S^* = \sum_{j=1}^{N_1} b_1 + \cdots + \sum_{j=1}^{N_n} b_n, \tag{11}$$

is a sum of Compound Negative Binomial distributed random variables.

2.2 Panjer's recursion

At first sight, determining the distribution function of the sum S^* is not a trivial task, as the random variables $\sum_{j=1}^{N_i} b_i$, $(i = 1, \ldots, n)$ are not mutually independent since they all are affected by the common mixing random variable Λ.

However, in this particular case one can prove that the distribution function of the combined portfolio is also Compound Negative Binomial distributed. Indeed, since the random variables N_i are conditionally independent we have that the moment generating function of S^* is given by

$$\begin{aligned}
m_{S^*}(t) &= E\left[e^{tS^*}\right] = E\left[\exp\left(t\sum_{i=1}^{n}\sum_{j=1}^{N_i} b_i\right)\right] \\
&= E\left[\prod_{i=1}^{n}\exp\left(t\sum_{j=1}^{N_i} b_i\right)\right] \\
&= E_\Lambda\left[E\left[\prod_{i=1}^{n}\exp\left(t\sum_{j=1}^{N_i} b_i\right) \mid \Lambda\right]\right] \\
&= E_\Lambda\left[\prod_{i=1}^{n} E\left[\exp\left(t\sum_{j=1}^{N_i} b_i\right) \mid \Lambda\right]\right] \\
&= E_\Lambda\left[\prod_{i=1}^{n} m_{N_i|\Lambda}\left(\ln m_{b_i}(t)\right)\right] \\
&= E_\Lambda\left[\prod_{i=1}^{n} \exp\left[q_i\ \Lambda\ \left(m_{b_i}(t)-1\right)\right]\right] \\
&= m_\Lambda\left[\sum_{i=1}^{n} q_i\ \left(m_{b_i}(t)-1\right)\right] \\
&= \left[\frac{\beta}{\beta-\sum_{i=1}^{n} q_i\ \left(m_{b_i}(t)-1\right)}\right]^{\alpha}. \qquad (12)
\end{aligned}$$

Now, let Y be a random variable with moment generating function given by

$$m_Y(t) = \frac{\sum_{i=1}^{n} q_i\ m_{b_i}(t)}{\sum_{i=1}^{n} q_i}. \qquad (13)$$

Then we find

$$\begin{aligned}
m_{S^*}(t) &= \left[\frac{\beta}{\beta-\ \sum_{i=1}^{n} q_i\left(m_Y(t)-1\right)}\right]^{\alpha} \\
&= \left[\frac{p}{1-(1-p)\ e^{\ln[m_Y(t)]}}\right]^{\alpha} \\
&= m_N\left(\ln m_Y(t)\right) \qquad (14)
\end{aligned}$$

with p given by

$$p = \frac{\beta}{\beta+\sum_{i=1}^{n} q_i}. \qquad (15)$$

Hence, we can conclude that S^* is Compound Negative Binomial distributed:

$$S^* = \sum_{i=1}^{N} Y_i \tag{16}$$

where N has a negative binomial distribution:

$$N \stackrel{d}{=} NB\left(\alpha, \frac{\beta}{\beta + \sum_{i=1}^{n} q_i}\right), \tag{17}$$

with mean and variance given by

$$E[N] = \frac{\alpha}{\beta} \sum_{i=1}^{n} q_i, \tag{18}$$

and

$$\mathrm{Var}[N] = \frac{\alpha}{\beta} \sum_{i=1}^{n} q_i \cdot \frac{\beta + \sum_{i=1}^{n} q_i}{\beta}, \tag{19}$$

respectively, and where the $Y_i \stackrel{d}{=} Y$ are i.i.d. and independent of N.

The distribution function of S^* can be computed using a well-known actuarial recursion, called "Panjer's recursion" (1981). It follows that

$$\Pr\left[S^* = 0\right] = \Pr\left[N = 0\right] \tag{20}$$

and also

$$\Pr\left[S^* = x\right] = \sum_{k=1}^{x} \left(a + \frac{bk}{x}\right) \Pr\left[Y = k\right] \Pr\left[S^* = x - k\right], \qquad (x = 1, 2, \ldots), \tag{21}$$

where

$$a = \frac{\sum_{i=1}^{m} q_{ik_i}}{\beta + \sum_{i=1}^{m} q_{ik_i}} \tag{22}$$

and

$$b = a\left(\beta - 1\right). \tag{23}$$

While Panjer's recursion is a mathematical beauty it must be pointed out that it sometimes suffers from numerical instability and in the literature several remedies have been suggested to deal with this issue. For instance, Gordy (2002) has suggested a saddle-point approximation whereas Giese (2003) and Haaf *et al.* (2003) have shown how Fast Fourier Transforms can be used to obtain the distribution function of S.

3 Incorporating direct dependence

3.1 Extension of the CreditRisk$^+$ model

As shown in Section 2 the standard one factor CreditRisk$^+$ model assumes that all the individual risks in the risky portfolio are affected by a common economic factor. Given the "state of the economy", the individual risks are assumed to be conditionally independent. In fact the three common approaches to model the credit risk of a portfolio all make this assumption of conditional independence; see Koyluoglu & Hickman (1998) for more details.

However in real situations the assumption of conditional independence is not always appropriate since some of the individual risks may be strongly connected to each other for structural reasons as well. For example, in a conglomerate the default of the strongest company may be contagious and irrespective of general economic situation trigger the default of all constituent companies; see also Giesecke and Weber (2006).

In order to account for this when estimating the distribution function of S.we will assume now that in the portfolio of n credit risks there are m ($m \leq n$) groups each of which contains at least one risk. The aggregate loss can now be written as

$$S = \sum_{i=1}^{m} \sum_{j=1}^{k_i} I_{ij}\ b_{ij} \tag{24}$$

with $k_i \geq 1, m \leq n$ and $\sum_{i=1}^{m} \sum_{j=1}^{k_i} = n$.

For convenience, we will assume that the risks are arranged in such way that $q_{i1} \leq \cdots \leq q_{ik_i}$ $(i = 1, ..., m)$, which means that in each group the risk with a lower index has a lower default probability. Moreover, we will assume that the risks within one group have the so-called "comonotonic property" which in a credit default context means that the default of a given credit risk will always give rise to the default of all risks with an equal or larger default probability. Such a "domino effect" assumption is quite appropriate in case of a group of risks where the default of the best rated company (the mother) also leads to the default of all other risks (the daughters). For a comprehensive overview on the concept of comonotonicity we refer to Dhaene *et al.* (2008).

In mathematical terms the comonotonicity assumption for the risks present within a given group amounts to

$$\Pr\left(I_{ij+l} = 1 \ \mid \ I_{ij} = 1\right) = 1 \qquad \left(j = 1, \cdots, k_i - 1;\ l = 1, \cdots, k_i - j\right). \tag{25}$$

From this it follows that

$$\sum_{j=1}^{k_i} I_{ij}\ b_{ij} = I_i C_i \tag{26}$$

with the probability density function (pdf) of the indicator r.v. I_i determined by:

$$\Pr\left[I_i = 1\right] = \max\left(q_{i1}, \cdots, q_{ik_i}\right) = q_{ik_i} \tag{27}$$

and the pdf of C_i given by

$$\Pr\left[C_i = \sum_{j=l}^{k_i}\ b_{ij}\right] = \frac{q_{il} - q_{il-1}}{q_{ik_i}}, \qquad l = 1, \cdots, k_i. \tag{28}$$

in which we tacitly assume that $q_{i0} = 0$. Hence, from (26) and (24) it follows that the aggregate loss S is again a sum of compound Bernoulli random variables

$$S = \sum_{i=1}^{m} I_i\ C_i, \tag{29}$$

and we denote $E\left(C_i\right) = c_i$ and $Var\left(C_i\right) = \sigma_{C_i}^2$. Following the reasoning from Section 2 we will then approximate the the distribution of S by the distribution function of S^* given as

$$S^* = \sum_{i=1}^{m}\sum_{j=1}^{N_i} C_{ij} \tag{30}$$

where the different C_{ij} are considered to be mutually independent, $C_{ij} \overset{d}{=} C_i$ and

$$N_i \overset{d}{=} NB\left(\alpha, \frac{\beta}{\beta + q_{ik_i}}\right) \tag{31}$$

It is important to observe that

$$\Pr\left[C_{ij} = \sum_{r=l}^{k_i}\ b_{ir} \mid\ \Lambda = \lambda\right] = \frac{q_{il}\lambda - q_{il-1}\lambda}{q_{ik_i}\lambda.} \tag{32}$$

$$= \frac{q_{il} - q_{il-1}}{q_{ik_i}}, \qquad l = 1, \cdots, k_i, \tag{33}$$

which effectively means that the random variables C_{ij} do not depend on the mixing conditioning random variable Λ so that the reasoning of Section 2 can be applied. We find that S^* can be written as

$$S^* = \sum_{i=1}^{N} Z_i \tag{34}$$

where

$$N \stackrel{d}{=} NB\left(\alpha, \frac{\beta}{\beta + \sum_{i=1}^{m} q_{ik_i}}\right), \tag{35}$$

and where the $Z_i \stackrel{d}{=} Z$ are i.i.d and independent of N with the moment generating function of the Z_i given by

$$m_Z(t) = \frac{\sum_{i=1}^{m} q_{ik_i}\ m_{C_i}(t)}{\sum_{i=1}^{m} q_{ik_i}}. \tag{36}$$

Hence, by applying Panjer's recursion, we have that:

$$\Pr\left[S^* = 0\right] = \Pr\left[N = 0\right] \tag{37}$$

and also

$$\Pr\left[S^* = x\right] = \sum_{k=1}^{x} \left(a + \frac{bk}{x}\right)\ \Pr\left[Z = k\right]\ \Pr\left[S^* = x - k\right], \qquad (x = 1, 2, \ldots), \tag{38}$$

where

$$a = \frac{\sum_{i=1}^{m} q_{ik_i}}{\beta + \sum_{i=1}^{m} q_{ik_i}} \tag{39}$$

and

$$b = a\left(\beta - 1\right). \tag{40}$$

3.2 Parameterisation

The parameters α and β have to be chosen such that the distribution functions of S and S^* are "as alike as possible".

A first natural requirement for the approximations to perform well is that the distribution functions of I_i and N_i are "as alike as possible". Therefore, we require that

$$\alpha = \beta, \tag{41}$$

For such choice it follows that:

$$\Pr\left[N_i = 0\right] = \left(\frac{\beta}{\beta + q_{ik_i}}\right)^{\beta} \approx 1 - q_{ik_i} = \Pr\left[I_i = 0\right], \tag{42}$$

while

$$\Pr\left[N_i = 1\right] = \beta \left(\frac{\beta}{\beta + q_{ik_i}}\right)^{\beta} (1 - \frac{\beta}{\beta + q_{ik_i}}) \approx q_{ik_i} = \Pr\left[I_i = 1\right] \tag{43}$$

so that the distributions of I_i and N_i will be "close" to each other, provided that $\frac{q_{ik_i}}{\beta}$ is small enough such that higher order terms can be neglected. Note that choosing $\alpha = \beta$ also implies that $E\left[N_i\right] = E\left[I_i\right] = q_{ik_i}$ so that $E\left[S\right] = E\left[S^*\right]$ will hold as well.

It remains to determine an explicit value for the parameter β. In order to fix this parameter we could resort to a study of the default statistics. Indeed, default rate volatility typically exhibits a strong relationship to the unconditional default rates and the coefficient of variation $v = \frac{\mathrm{Stdev}(N)}{\mathrm{E}(N)}$ appears to be a meaningful statistic in this context. For example, Carty & Lieberman (1995) have studied a large database covering "All Corporate" default experience from 1970-1995 and reported a coefficient of variation $v = 0.78$.

Next, for large homogeneous portfolios we find from the expressions (18) and (19) that v is approximately given by

$$\beta \approx \frac{1}{v^2}, \tag{44}$$

so that β can be estimated from default statistics directly.

When the variance of S is assumed to be known we can also derive β by requiring that $Var\left(S^*\right) = Var\left(S\right).$ First we remark that for $i \neq j$ we have that

$$\begin{aligned} Cov\left[\sum_{k=1}^{N_i} C_{ik}\,, \sum_{l=1}^{N_j} C_{jl}\right] &= E\left[Cov\left[\sum_{k=1}^{N_i} C_{ik}\,, \sum_{l=1}^{N_j} C_{jl}\right] \mid N_i,\ N_j\right] \\ &\quad + Cov\left[E\left[\sum_{k=1}^{N_i} C_{ik}\,, \sum_{l=1}^{N_j} C_{jl}\right] \mid N_i,\ N_j\right] \\ &= 0 + c_i c_j Cov\left[N_i, N_j\right], \end{aligned} \tag{45}$$

where

$$\begin{aligned} Cov\left[N_i\ , N_j\right] &= E\left[E\left[N_i\ N_j \mid \Lambda\right]\right] - E\left[N_i\right]\ E\left[N_j\right] \\ &= E\left[E\left[N_i \mid \Lambda\right]\ E\left[N_j \mid \Lambda\right]\right] - q_{ik_i}\ q_{jk_j} \\ &= q_{ik_i}\ q_{jk_j} \left\{E\left[\Lambda^2\right] - 1\right\} \\ &= q_{ik_i}\ q_{jk_j}\ Var\left[\Lambda\right] \\ &= \frac{q_{ik_i}\ q_{jk_j}}{\beta}, \end{aligned} \tag{46}$$

When $i = j$ we have that

$$\begin{aligned} Var\left[\sum_{k=1}^{N_i} C_{ik}\right] &= E\left[Var\left[\sum_{k=1}^{N_i} C_{ik}\ , \sum_{l=1}^{N_i} C_{il}\right] \mid N_i\right] \\ &\quad + Var\left[E\left[\sum_{k=1}^{N_i} C_{ik}\ , \sum_{l=1}^{N_i} C_{il}\right] \mid N_i\right] \\ &= E[N_i]\sigma_{C_i}^2 + c_i^2 Var[N_i]. \end{aligned}$$

with

$$E\left[N_i\right] = q_{ik_i}, \tag{47}$$

and

$$Var\left[N_i\right] = q_{ik_i}\left(1 + \frac{q_{ik_i}}{\beta}\right), \tag{48}$$

respectively, so that the variance of S^* will be given by

$$\begin{aligned} Var\left[S^*\right] &= \sum_{i=1}^{m} Var\left[\sum_{k=1}^{N_i} C_{ik}\right] + 2\sum_{i=1}^{m}\sum_{j=i+1}^{m} Cov\left[\sum_{k=1}^{N_i} C_{ik}\ , \sum_{l=1}^{N_j} C_{jl}\right] \\ &= \frac{1}{\beta}\sum_{i=1}^{m}\left(c_i\ q_{ik_i}\right)^2 + \left(\sum_{i=1}^{m}\left(\sigma_{C_i}^2 + c_i^2\right)\ q_{ik_i}\right) + \frac{2}{\beta}\sum_{i=1}^{m}\sum_{j=i+1}^{m} c_i c_j q_{ik_i} q_{jk_j} \\ &= \frac{\left(\sum_{i=1}^{m} c_i\ q_{ik_i}\right)^2}{\beta} + \left(\sum_{i=1}^{m}\left(\sigma_{C_i}^2 + c_i^2\right)\ q_{ik_i}\right). \end{aligned} \tag{49}$$

Hence, the condition $Var\left(S\right) = Var\left(S^*\right)$ will be fulfilled if β is chosen as follows:

$$\beta = \frac{\left(\sum_{i=1}^{m} c_i\ q_i\right)^2}{Var(S) - \left(\sum_{i=1}^{m} \left(\sigma_{C_i}^2 + c_i^2\right)\ q_{ik_i}\right)}. \tag{50}$$

Note that in order to guarantee that $\beta > 0$ we must have

$$var(S) > \sum_{i=1}^{m} \left(\sigma_{C_i}^2 + c_i^2\right)\ q_{ik_i} > \sum_{i=1}^{m} \sigma_{C_i}^2\ q_{ik_i} + \sum_{i=1}^{n} c_i^2\ q_i(1 - q_{ik_i}) = Var(S_{ind}), \tag{51}$$

where S_{ind} is denoting the aggregate claims amount assuming that all risks in the sum S are independent. Hence, the one factor CreditRisk$^+$ model only makes sense in the case that the different correlations are (on average) positive and this condition will in practice always be fulfilled. Let us also observe that for $i \neq j$, the pairwise correlations in the approximated model are given by

$$\begin{aligned} corr\,[N_i\ , N_j] &= \frac{\sqrt{q_{ik_i}\ q_{jk_j}}}{\beta} \frac{1}{\sqrt{1 + \frac{q_{ik_i}}{\beta}}\sqrt{1 + \frac{q_{jk_j}}{\beta}}} \\ &\approx \frac{\sqrt{q_{ik_i}\ q_{jk_j}}}{\beta} \end{aligned}$$

so that in the model we propose, in fact we replace the known correlations $corr\,[I_i\ , I_j]$ by $corr\,[N_i,\ N_j] \approx \frac{\sqrt{q_i\ q_j}}{\beta}$ for $i \neq j$. Hence, the CreditRisk$^+$ model will perform the best if the "exact" correlations $corr\,[I_i\ , I_j]$ are approximately equal to $\frac{\sqrt{q_i\ q_j}}{\beta}$.

4 Numerical examples

We consider two examples.

Firstly we consider the example on page 59 of the original CreditRisk$^+$ documentation (Credit Suisse Financial Products (1997)). They considered 25 clients with exposures between 358,475 and 20,238,895, default rates between 1.5% and 30% and $\beta = 4$. In the base case, we assume in line with the original CreditRisk$^+$ no direct (intra-)group dependence between the different clients. In the second case, we assume the largest two companies are in the same group. It is our experience that this may be the case with real-life portfolios. To avoid possible numerical instabilities we have not used Panjer's original recursion but applied the techniques from Giese (2003); see also Haaf et al. (2003). We present the different quantiles in Table 1.

	no direct dependence	*direct dependence*
0.75-quantile	20.53	18.74
0.90-quantile	31.42	33.35
0.99-quantile	55.24	64.65
0.995-quantile	61.93	74.31

Table 1: Various quantiles of the aggregate loss distribution for the first example with and without direct (intra-)group dependence. The unit is 1 million.

This example shows the significant impact that group dependence can have, particularly on the upper quantiles of the loss distribution. The 99.5% quantile increases by approximately 25% when group dependence is included. It may be surprising to see that assuming (intra-)group dependence does not seem to imply that all quantiles increase as compared to the base case. Indeed, assuming (intra-)group dependence does not change the expectation of the loss variable at hand so that the distribution functions must cross at least once; see also Dhaene *et al.* (2008).

The second example involves a portfolio of 10,000 clients. There are 4000 "small" clients with exposure of 1 and a default probability of 1%, 4000 "medium" clients with exposure of 2 and a default probability of 0.5% and 2000 "large" clients with exposure of 4 and a default probability of 0.25%. In the first case we assume no structural dependence and in the second we assume each "large" company is in a group with one "small" company and one "medium" company. Various quantiles are presented in Table 2.

Also in this example we see that group dependence has a significant impact on the tail of the loss distribution.

	no direct dependence	*direct dependence*
0.75-quantile	105	131
0.90-quantile	138	173
0.99-quantile	208	263
0.995-quantile	228	288

Table 2: Various quantiles of the aggregate loss distribution with and without direct (intra-)group dependence.

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